Control Theory I

By

Assist. Prof. Dr.

Yousif Ismail Al Mashhadnay

2019 – 2020

No	Subject	Week (3 H/W)
1.	**Introduction to control system:** ➢ *Introduction.* ➢ *Open Loop System.* ➢ *Close Loop System.* ➢ *Definitions.* ➢ *The engineering control problem.*	1

2.	**Mathematical Representation of physical systems:** ➢ ***Linear system, non linear system.*** ➢ ***transfer functions, Block diagram.*** ➢ ***electrical systems.*** ➢ ***mechanical translation system.*** ➢ ***mechanical rotational system.*** ➢ ***Thermal system.*** ➢ ***Modeling in state space.*** ➢ ***How to derive transfer function from the state space equations.*** ➢ ***State space representation of dynamic system.***	3

3.	**Block diagrams Processing:** ➢ ***Procedures for drawing a block diagram.*** ➢ ***block diagram reduction.*** ➢ ***closed loop system subjected to a disturbance.*** ➢ ***multivariable Systems, transfer matrices.***	**2**
4.	**Signal flow graphs:** ➢ ***Signal flow graph representation of linear system.*** ➢ ***Mason's gains formula for signal flow graph.*** ➢ ***Transfer function process in signal flow graph.***	**2**

5.	**Transient response analysis:** ➢ ***Test signals, impulse response function.*** ➢ ***first order system, higher order system.*** ➢ ***definitions of time constant, damping ratio and natural frequency.*** ➢ ***definitions of transient response specifications.*** ➢ ***impulse response, dominant poles.***	2

6.	**Steady – state error in unity- feedback control system:** ➢ ***Classifications of control systems.*** ➢ ***static position error coefficients.*** ➢ ***dynamic error coefficients.***	**2**
7.	***Routh's Stability Criterion*** ➢ ***Introduction.*** ➢ ***Routh's Criteria Rules.*** ➢ ***Solved problem for Checking System Stability.***	**2**

8.	**Root Locus:** ➢ *Root locus plot.* ➢ *general rules for constructing root loci.* ➢ *special cases, conditionally stable system.* ➢ *non-minimum phase systems.*	1

References:

1) *" Linear Control System Analysis and Design with MATLAB " by John J. D'Azzo and Constantine H. Houpis , 2003.*

2) *"Classical Control Theory", By Yousif Al Mashhadany, first Edition, 2014*

3) *"Modern Control Engineering" , By Katsuhiko Ogata, Fifth Edition, 2010*

4) *"Control Systems Engineering" , Sixth Edition, By Norman S. Nise, 2011*

University of Anbar
College of Engineering
Dept. of Electrical Engineering

Control Theory I
Assist. Prof. Dr. Yousif Al Mashhadany
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Lecture No. One

Introduction To Control System

This Lecture will discuss the following subjects:

1) **This lecture discusses the following topics :**
2) **Introduction.**
3) **Open Loop System**
4) **Close Loop System.**
5) **Definitions of control system.**
6) **The engineering control problem.**
7) **Solved Examples**
8) **Problems**

1.1. Introduction:

The toaster in Fig.1.1 can be set for the desired darkness of the toasted bread. The setting of the "darkness" knob, or timer, represents the input quantity, and the degree of darkness and crispness of the toast produced is the output quantity. If the degree of darkness is not satisfactory, because of the condition of the bread or some similar reason, this condition can in no way automatically alter the length of time that heat is applied. Since the output quantity has no influence on the input quantity, there is no feedback in this system. The heater portion of the toaster represents the dynamic part of the overall system, and the timer unit is the reference selector.

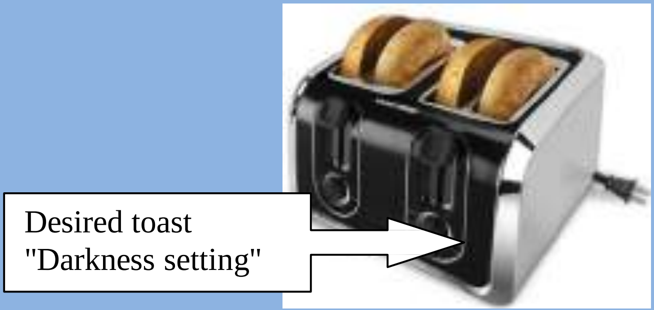
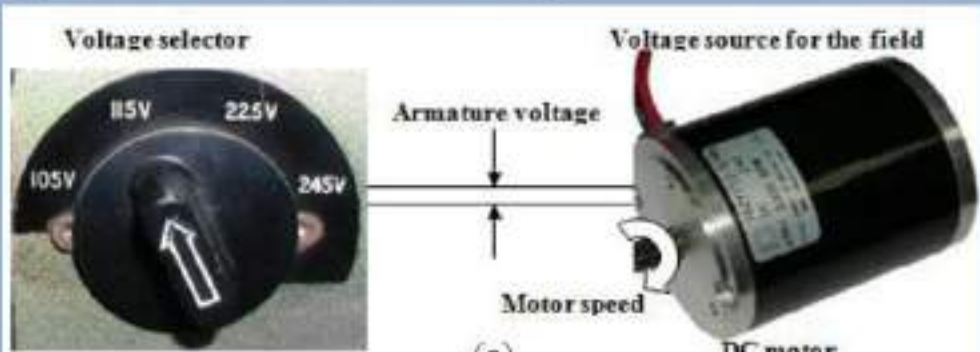


Fig. 1.1 Open-loop control system automatic toaster

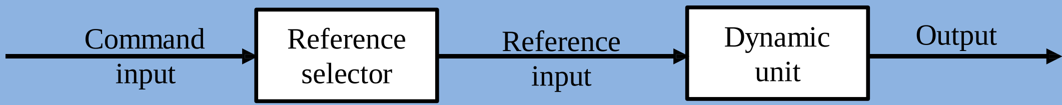
The dc shunt motor of Fig. 1.2 is another example. For a given value of field current, a required value of voltage is applied to the armature to produce the desired value of motor speed. In this case the motor is the dynamic part of the system, the applied armature voltage is the input quantity, and the speed of the shaft is the output quantity. A variation of the speed from the desired value, due to a change of mechanical load on the shaft, can in no way cause a change in the value of the applied armature voltage to maintain the desired speed. Therefore, the output quantity has no influence on the input quantity.

1.2. Open Loop System:

Systems in which the output quantity has no effect upon the input quantity are called open-loop control systems. The examples just cited are represented symbolically by a functional block diagram, as shown in Fig. 1.2.C. In this figure, (1) the desired darkness of the toast or the desired speed of the motor is the command input, (2) the selection of the value of time on the toaster timer or the value of voltage applied to the motor armature is represented by the reference-selector block, and (3) the output of this block is identified as the reference input. The reference input is applied to the dynamic unit that performs the desired control function, and the output of this block is the desired output.



(a)



(b)

Fig. 1.2. Open-loop control system (a) electric motor; (b) functional block diagram.

A person could be assigned the task of sensing the actual value of the output and comparing it with the command input. If the output does not have the desired value, the person can alter the reference-selector position to achieve this value. Introducing the person provides a means through which the output is feedback and is compared with the input. Any necessary change is then made in order to cause the output to equal the desired value.

1.3. Close Loop System:

The feedback action therefore controls the input to the dynamic unit. Systems in which the output has a direct effect upon the input quantity are called closed loop control systems. To improve the performance of the closed-loop system so that the output quantity is as close as possible to the desired quantity, the person can be replaced by a mechanical, electrical, or other form of a comparison unit. The functional block diagram of a single-input single-output (SISO) closed-loop control system is illustrated in Fig. 1.3. Comparison between the reference input and the feedback signals results in an actuating signal that is the difference between these two quantities. The actuating signal acts to maintain the output at the desired value. This system is called a closed-loop control system..

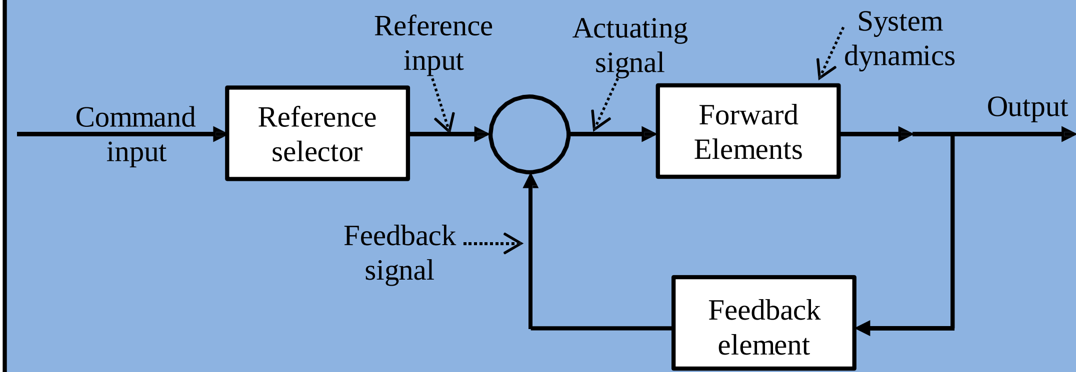


Fig. 1.3. Functional block diagram of a closed-loop system

The designation closed-loop implies the action resulting from the comparison between the output and input quantities in order to maintain the output at the desired value. Thus, the output is controlled in order to achieve the desired value.

Examples of closed-loop control systems are illustrated in Figs. 1.4&1.5 . In a home heating system the desired room temperature (command input) is set on the thermostat in Fig.1.4. (reference selector).A bimetallic coil in the thermostat is affected by both the actual room temperature (output) and the reference-selector setting. If the room temperature is lower than the desired temperature, the coil strip alters its shape and causes a mercury switch to operate a relay, which turns on the furnace to produce heat in the room.

When the room temperature reaches the desired temperature, the shape of the coil strip is again altered so that the mercury switch opens. This deactivates the relay and in turn shuts off the furnace. In this example, the bimetallic coil performs the function of a comparator since the output (room temperature) is fed back directly to the comparator. The switch, relay, and furnace are the dynamic elements of this closed-loop control system.

A closed-loop control system of great importance to all multistory buildings is the automatic elevator of Fig.1.5. A person in the elevator presses the button corresponding to the desired floor. This produces an actuating signal that indicates the desired floor and turns on the motor that raises or lowers the elevator. As the elevator approaches the desired floor, the actuating signal decreases in value and, with the proper switching sequences, the elevator stops at the desired floor and the actuating signal is reset to zero. The closed loop control system for the express elevator in the Sears Tower building in Chicago is designed so that it ascends or descends the 103 floors in just under1min with maximum passenger comfort.

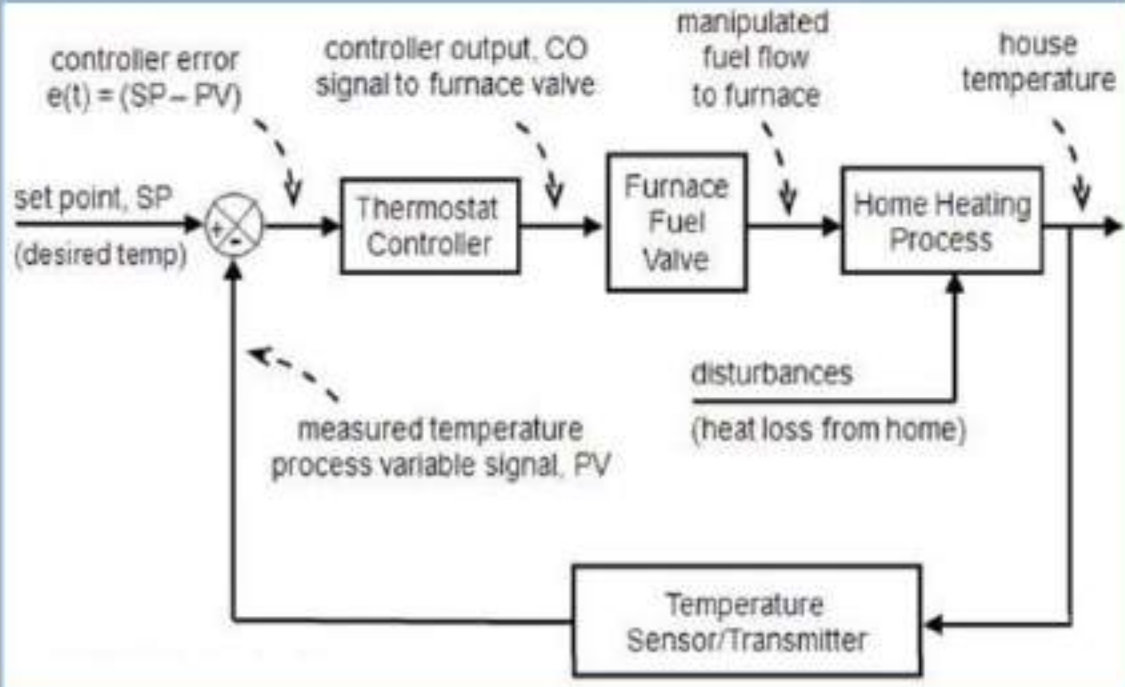


Fig.1.4. Home heating block diagram control system.

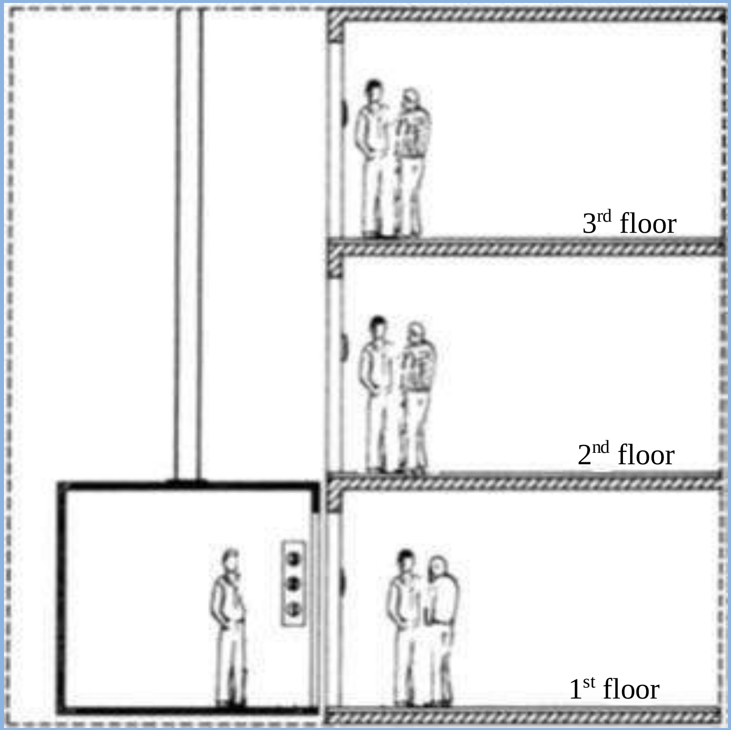


Fig. 1.5. Automatic elevator.

1.4. Definitions of control system.

From the above description there are many terms can be defined as:

Control Theory: It is that part of science which concern control problems.

Control Problem: If we want something to act or vary according to a certain performance specification, then we say that we have a control problem. Ex. We want to keep the temperature in a room at certain level and as we order, then we say that we have temperature control problem.

Plant: A piece of equipment's the purpose of which is to perform a particular operation (we will call any object to be controlled a plant). Ex. Heating furnace, chemical reactor or space craft.

The system: A combination of components that act together to perform a function not possible with any of the individual parts. The word system as used herein is interpreted to include physical, biological, organizational, and other entities, and combinations thereof, which can be represented through a common mathematical symbolism. The formal name systems engineering can also be assigned to this definition of the word system. Thus, the study of feedback control systems is essentially a study of an important aspect of systems engineering and its application.

Process: Progressively continuing operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead towered a particular results or end. In this lectures we call any operation to be controlled a process.

Reference selector (reference input element): The unit that establishes the value of the reference input. The reference selector is calibrated in terms of the desired value of the system output.

Reference input: The reference signal produced by the reference selector, i.e., the command expressed in a form directly usable by the system. It is the actual signal input to the control system.

Disturbance input: An external disturbance input signal to the system that has an unwanted effect on the system output.

Forward element (system dynamics): The unit that reacts to an actuating signal to produce a desired output. This unit does the work of controlling the output and thus may be a power amplifier.

Output (controlled variable): The quantity that must be maintained at a prescribed value, i.e., following the command input without responding the disturbance inputs.

Open-loop control system: A system in which the output has no effect upon the input signal. Ex. heater, light, washing machine.

Feedback element: The unit that provides the means for feeding back the output quantity, or a function of the output, in order to compare it with the reference input.

Actuating signal: The signal that is the difference between the reference input and the feedback signal. It is the input to the control unit that causes the output to have the desired value.

Closed-loop control system: A system in which the output has an effect upon the input quantity in such a manner as to maintain the desired output value.

The fundamental difference between the open- and closed-loop systems is the feedback action, which may be continuous or discontinuous. In one form of discontinuous control the input and output quantities are periodically sampled and discontinuous. Continuous control implies that the output is continuously feedback and compared with the reference input compared; i.e., the control action is discontinuous in time. This is commonly called a digital, discrete-data or sampled-data feedback control system. A discrete data control system may incorporate a digital computer that improves the performance achievable by the system. In another form of discontinuous control system the actuating signal must reach a prescribed value before the system dynamics reacts to it; i.e., the control action is discontinuous in amplitude rather than in time. This type of discontinuous control system is commonly called an on-off or relay feedback control system. Both forms

may be present in a system. In this text continuous control systems are considered in detail since they lend themselves readily to a basic understanding of feedback control systems. With the above introductory material, it is proper to state a definition of a feedback control system: "A control system that operates to achieve prescribed relationships between selected system variables by comparing functions of these variables and using the comparison to effect control." The following definitions are also used.

Servomechanism (often abbreviated as servo): The term is often used to refer to a mechanical system in which the steady-state error is zero for a constant input signal. Sometimes, by generalization, it is used to refer to any feedback control system.

Regulator: This term is used to refer to systems in which there is a constant steady-state output for a constant signal. The name is derived from the early speed and voltage controls, called speed and voltage regulators.

Major advantages of open loop control system:

1. Simple construction and ease of maintenance.
2. Less expensive than the corresponding closed loop system.
3. There is no stability problem.
4. Convenient when output is hard to measure or economically not feasible.

The disadvantages of open loop control systems are as follows:

1. Disturbances and changes in calibration cause errors and the output may be different from what is desired.
2. To maintain the required quality in the output, recalibration is necessary from time to time.

5. The engineering control problem:

In general, a control problem can be divided into the following steps:

1. A set of performance specifications is established.
2. The performance specifications establish the control problem.

3. A set of linear differential equations that describe the physical system is formulated or a system identification technique is applied in order to obtain the plant model transfer functions.
4. A control-theory design approach, aided by available computer aided-design (CAD) packages or specially written computer programs, involves the following:
    a. The performance of the basic (original or uncompensated) system is determined by application of one of the available methods of analysis (or a combination of them).

b. If the performance of the original system does not meet the required specifications, a control design method is selected that will improve the system's response.

c. For plants having structured parameter uncertainty, the quantitative feedback theory (QFT) design technique may be used. Parametric uncertainty is present when parameters of the plant to be controlled vary during its operation.

5. A simulation of the designed nonlinear system is performed.
6. The actual system is implemented and tested.

Design of the system to obtain the desired performance is the control problem. The necessary basic equipment is then assembled into a system to perform the desired control function. Although most systems are nonlinear, in many cases the nonlinearity is small enough to be neglected, or the limits of operation are small enough to allow a linear analysis to be used. This textbook considers only linear systems. A basic system has the minimum amount of equipment necessary to accomplish the control function. After a control system is synthesized to achieve the desired performance, final adjustments can be made in a simulation, or on the actual system, to take into account the nonlinearities that were neglected. A computer is generally used in the design, depending upon the complexity of the system. The essential aspects of the control system design process are illustrated in Fig.1.6. Note: The development of this figure is based upon the application of the QFT design technique. A similar figure may be developed for other design techniques. The intent of (Fig.1.6) is to give the reader an overview of what is involved in achieving a successful and practical control system design.

Finally the following design policy includes factors that are worthy of consideration in the control system design problem:

1. Use proven design methods.
2. Select the system design that has the minimum complexity.

3. Use minimum specifications or requirements that yield a satisfactory system response. Compare the cost with the performance and select the fully justified system implementation.
4. Perform a complete and adequate simulation and testing of the system.

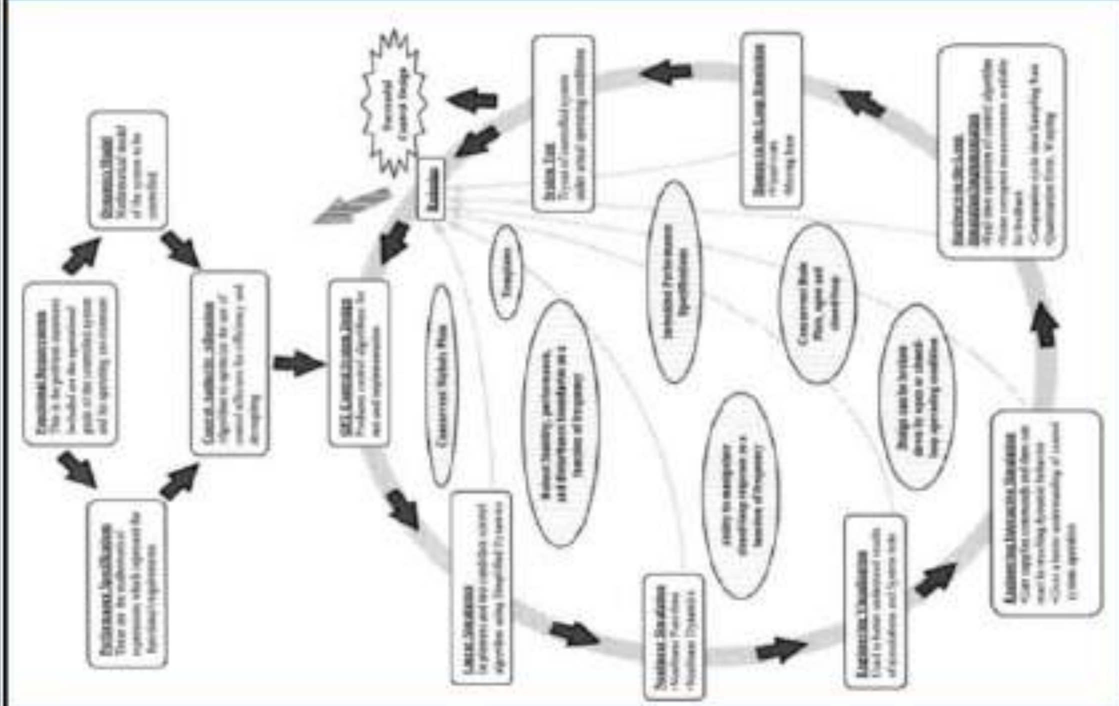


Fig. 1.6. A control system design process: bridging the gap [ref 2].

1.6. Solved Examples

Example 1: Fig.1.7.a. is a schematic diagram of a liquid level control system. Here the automatic controller maintains the liquid level by comparing the actual level with the desired level and correcting any error by adjusting the opening of pneumatic valve. Fig.1.7.b. is the corresponding block diagram of the control system.

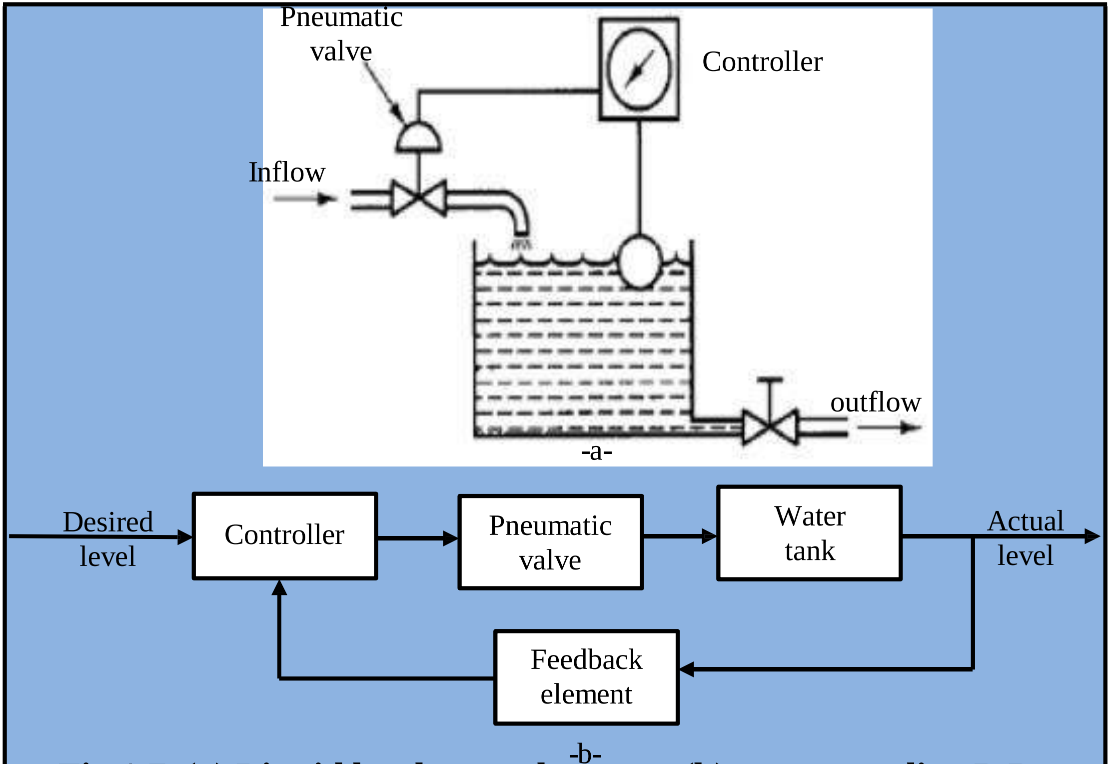


Fig.1.7. (a) Liquid level control system. (b) corresponding B.D.

To draw the block diagram for a human operated liquid level as an example of closed loop control system we will need the following parts (see Fig. 1.8):

1. Eyes as a sensor . 2. Brain as a controller
3. Muscles as a pneumatic valve as in the following block diagram

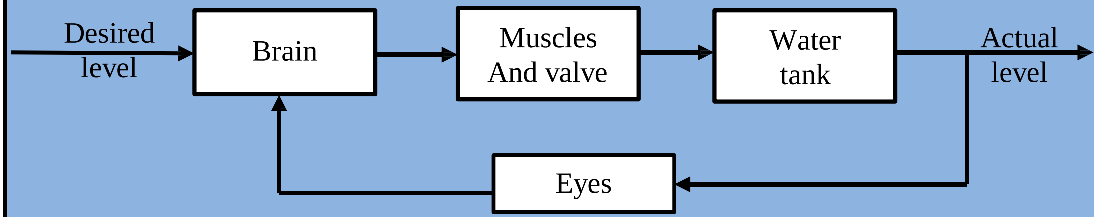


Fig.1.8. Block diagram of human operated liquid level control system

Example 2. An engineering organization system is composed of major groups such as management, research and development, preliminary design, experiment, product design and drafting, fabrication and assembling and testing. These groups are interconnected to make up the whole operation. The system may be organized by reducing it to the most elementary set of components necessary that can provide the analytical detail required and by representing the dynamic characteristics of each component by a set of simple equations. The functional block diagram of the engineering organization can be illustrated as in the block diagram is shown in Fig. 1.9.

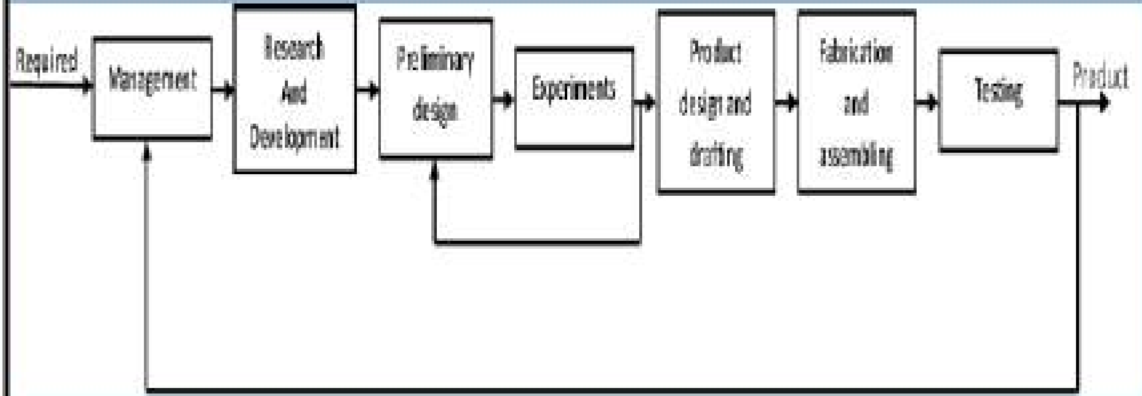


Fig.1.9. Block Diagram of an engineering organization system

1.7. Problems

1. Give two examples of feedback control systems in which a human acts as a controller?
2. Explain the open-loop control system by functional diagram and describe the blocks by practical example?
3. Explain the closed-loop system by functional block diagram and compared it with open-loop control system?
4. Many closed-loop and open-loop control systems may be found in homes. List several examples and describe them?
5. Draw the general block diagram of control system and explain each block in the sketch?

University of Anbar
College of Engineering
Dept. of Electrical Engineering

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Lecture No. Two

Mathematical Representation of Physical Systems

This lecture discusses the following topics :

1.Introduction:

2. Electrical system.

3. Multiloop Electric Circuits.

4. State Space Concepts (S.S)

5. Transfer Function (T.F):

6.Correlation between transfer functions

and state-space equations.

7. **Transfer Function From State-Variable Representation.**

8. **State Variable Representation From Transfer Function.**

9. **Properties of the State Transition Matrix:**

10. **Complex impedances.**

11. **Transfer function of nonloading cascaded system.**

12. **Mecanical Systems.**

    2.12.1. Translational mechanical systems

    2.12.2. Rotational mechanical systems

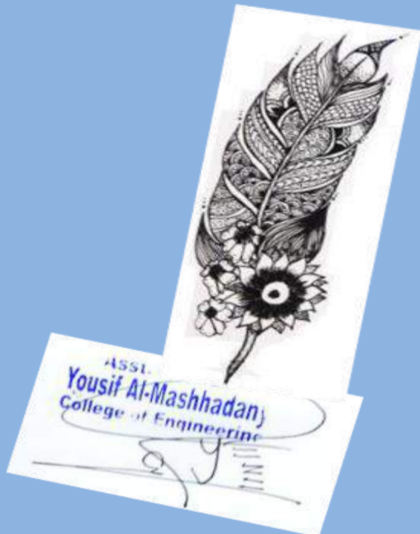
13. Liquid systems.

14. Thermal systems.

15. Extra systems

- **2.15.1. Gear trains**
- **2.15.2. Potentiometer**
- **2.15.3. Error Detector**
- **2.15.4. First-Order Op-Amp**

2.16. Simulation diagram



2.1. Introduction:

● Mechanical, electrical, thermal, hydraulic, economic, biological, etc, systems, may be characterized by differential equations.

● The response of dynamic system to an input may be obtained if these differential equations are solved.

● The differential equations can be obtained by utilizing physical laws governing a particular system, for example, Newton's laws for mechanical systems, Kirchhoff's laws for electrical systems, etc.

Mathematical models: The mathematical description of the dynamic characteristic of a system. The first step in the analysis of dynamic system is to derive its model. Models may assume different forms, depending on the particular system and the circumstances. In obtaining a model, we must make a compromise between the simplicity of the model and the accuracy of results of the analysis.

Transfer functions: The transfer function of a linear time-invariant system is define to be the ratio of the Laplace transform (z-transform for sampled data systems) of the output to the Laplace transform of the input (driving function), under the assumption that all initial conditions are zero.

Example: Consider the linear time-invariant system

$$a_o y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y^{\cdot} + a_n y = b_o x^{(m)} + b_1 x^{(m-1)} + ... + b_{m-1} x^{\cdot} + b_m x \ , n \geq m$$

Taking the Laplace transform of y(t)

$$\ell(a_o y^{(n)}) = a_o \ell(y^{(n)}) = a_o [S^n Y(S) - S^{n-1} y(0) - S^{n-2} y'(0) - ...... - y^{(n-1)}(0)$$

$$\ell(a_1 y^{(n-1)}) = a_1 \ell(y^{(n-1)}) = a_o [S^{n-1} Y(S) - S^{n-2} y(0) - S^{n-3} y'(0) - ...... - y^{(n-2)}(0)$$

$$\ell(a_2 y^{(n-2)}) = a_1 \ell(y^{(n-2)}) = a_o [S^{n-2} Y(S) - S^{n-3} y(0) - S^{n-4} y'(0) - ...... - y^{(n-3)}(0)$$

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$$\ell(a_n y) = a_n \ell(y) = a_n Y(s)$$

same thing is applied to obtain the L.T. of x(t).by substitute all initial condition to zero. The transfer function of the system become.

$$\text{Transfer function} = G(s) = \frac{b_o S^m + b_1 S^{(m-1)} + \dots\dots + b_{m-1} S + b_m}{a_o S^n + a_1 S^{(n-1)} + \dots\dots + a_{n-1} S + a_m}$$

●Transfer function is not provide any information concerning the physical structure of the system (the T.F. of many physically different system can be identical).

●The highest power of s in the denominator of T. F. is equal to the order of the highest derivative term of the output. If the highest power of s is equal to n the system is called an nth order system.

How you can obtain the transfer function (T. F.)?

1- Write the differential equation of the system

2Take the L. T. of the differential equation, assuming all initial condition to be zero.

3 Take the ratio of the output to the input. This ratio is the T. F.

In general, variables that are functions of time are represented by lowercase letters. These are sometimes indicated by the form x(t), but more often this is written just as x. There are some exceptions, because of established convention, in the use of certain symbols. To simplify the writing of differential equations. The symbols D and1/D are defined by:

$$Dy \equiv \frac{dy(t)}{dt}, \quad D^2 y \equiv \frac{d^2 y(t)}{dt^2}$$

$$D^{-1} y \equiv \frac{1}{D} y \equiv \int_0^t y(\tau)\, d\tau + \int_{-\infty}^{0} y(\tau)\, d\tau = \int_0^t y(\tau)\, d\tau + Y_0$$

where *Yo* represents the value of the integral at time t = 0, that is, the initial value of the integral.

2.2. Electrical system:

For the series RLC circuit shown in Fig.2.1,

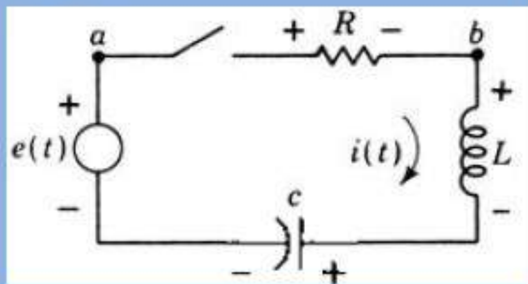


Fig.2.1. Series Resistor–Inductor–Capacitor Circuit

the applied voltage is equal to the sum of the voltage drops when the switch is closed:

$$V_L + V_R + V_C = e$$

(2.4)

$$LD_i + Ri + \frac{1}{CD} i = e$$

(2.5)

The circuit equation can be written in terms of the voltage drop across any circuit element. For example, in terms of the voltage across the resistor, *VR* = *Ri*, the equation become:

$$\frac{L}{R}D_{VR} + v_R + \frac{1}{RCD}v_R = e$$

(2.6)

For LRC circuit in Fig.2.2.

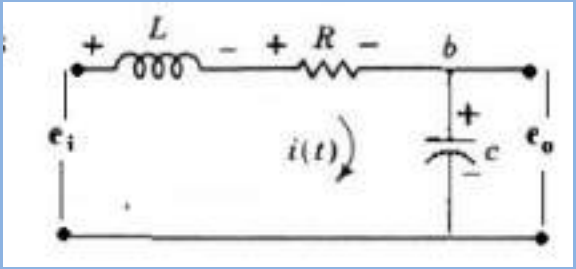


Fig.2.2. Series Resistor–Inductor–Capacitor Circuit

Applying Kirchhoff's voltage law to the system shown. We obtain the following equation;

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

(2.7)

$$\frac{1}{C}\int i\,dt = e_o$$

(2.8)

Above two equations give a mathematical model of the circuit. Taking the L.T. of equations, assuming zero initial conditions, we obtain:

$$LSI(S) + RI(S) + \frac{1}{CS} I(S) = E_i(S)$$

$$\frac{1}{CS} I(S) = E_o(S)$$

The final transfer function of the series RLC circuit will be as in the following equation :

$$\frac{E_o(S)}{E_i(S} = \frac{1}{LCS^2 + RCS + 1}$$

2.3. Multi loop Electric Circuits

Multi loop electric circuits (see Fig.2.3) can be solved by either loop or nodal equations.

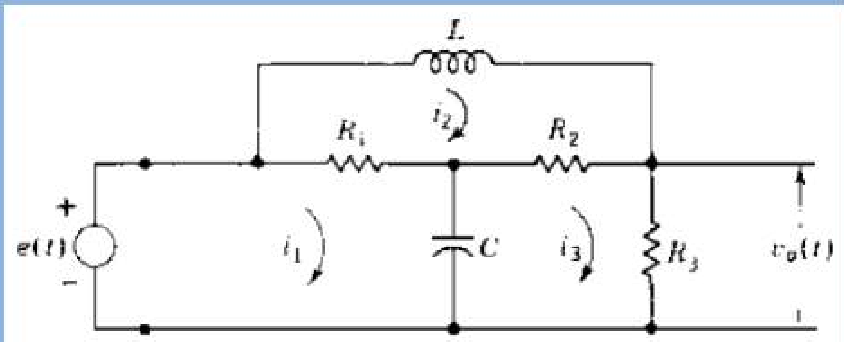


Fig.2.3. Multi loop network

The following example illustrates both methods. The problem is to solve for the output voltage Vo.

Loop Method: A loop current is drawn in each closed loop (usually in a clockwise direction); then Kirchhoff 's voltage equation is written for each loop:

$$(R_1 + \frac{1}{CD})i_1 - R_1 i_2 - \frac{1}{CD} i_3 = e$$

$$-R_1 i_1 + (R_1 + R_2 + LD)i_2 - R_2 i_3 = 0$$

$$-\frac{1}{CD} i_1 - R_2 i_2 + (R_2 + R_3 + \frac{1}{CD})i_3 = 0$$

The output voltage is Vo=R3 i3 These four equations must be solved simultaneously to obtain Vo(t) in terms of the input voltage e(t) and the circuit parameters.

Node Method: The junctions, or nodes, are labeled by letters in Fig.2.4. Kirchhoff's current equations are written for each node in terms of the node voltages, where node **d** is taken as reference. The voltage $\mathbf{V_{bd}}$ is the voltage of node **b** with reference to node **d**. For simplicity, the voltage $\mathbf{V_{bd}}$ is written just as $\mathbf{V_{b}}$.

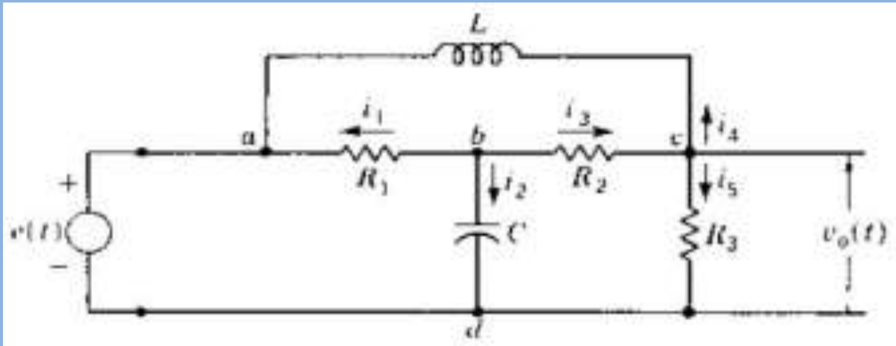


Fig.2.4. Multi node network

$$i_1 + i_2 + i_3 = 0$$

$$-i_3 + i_4 + i_5 = 0$$

Since there is one known node voltage ***Va=e*** two unknown voltages ***Vb*** and ***Vo***, only two equations are required:

For node b and node c:

In terms of the node voltages, these equations are:

$$\frac{v_b - v_a}{R_1} + CDV_b + \frac{v_b - v_0}{R_2} = 0$$

$$\frac{v_0 - v_b}{R_2} + \frac{V_o}{R_3}V + \frac{1}{LD}(v_o - e) = 0$$

Rearranging the terms in order to systematize the form of the equations gives:

$$\left(\frac{1}{R_1} + CD + \frac{1}{R_2}\right)V_b + \frac{v_0}{R_2} = \frac{e}{R_1}$$

$$\frac{1}{R_1}V_b + \left(\frac{1}{LD} + \frac{1}{R_2} + \frac{1}{R_3}\right)V_0 = \frac{e}{LD}$$

For this example, only two nodal equations are needed to solve for the potential at node *c*. An additional equation must be used if the current in *R3* is required. With the loop method, three equations must be solved simultaneously to obtain the current in any branch; an additional equation must be used if the voltage across *R3* is required. The method that requires the solution of the fewest equations should be used. This varies with the circuit.

The rules for writing the node equations are summarized as follows:

1.The number of equations required is equal to the number of unknown node voltages.

2. An equation is written for each node.

3. Each equation includes the following:

(a) The node voltage multiplied by the sum of all the admittances that are connected to this node. This term is positive.

(b) The node voltage at the other end of each branch multiplied by the admittance connected between the two nodes. This term is negative.

2.4. State Space Concepts:

Basic matrix properties are used to introduce the concept of state and the method of writing and solving the state equations.

State: The state of a system is a mathematical structure containing a set of n variables $x1(t)$, $x2(t)$, . . . , $xi(t)$, . . . , $xn(t)$, called the state variables, such that the initial values $xi(to)$ of this set and the system inputs $uj(t)$ are sufficient to describe uniquely the system's future response of $t \geq to$. A minimum set of state variables is required to represent the system accurately. The m inputs, $u1(t)$, $u2(t), \ldots, uj(t), \ldots, um(t)$, are deterministic; i.e., they have specific values for all values of time $t \geq to$.

Generally the initial starting time *to* is taken to be zero. The state variables need not be physically observable and measurable quantities; they may be purely mathematical quantities. The following additional definitions apply:

State Vector: The set of state variables *xi(t)* represents the elements or components of the n-dimensional state vector *x(t);* that is,

$$X(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ . \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix} \equiv X$$

The order of the system characteristic equation is **n**, and the state equation representation of the system consists of n first-order differential equations. When all the inputs **uj (t)** to a given system are specified for **t> to**, the resulting state vector uniquely determines the system behavior for any **t > to**.

State Space: State space is defined as the n-dimensional space in which the components of the state vector represent its coordinate axes.

State Trajectory: State trajectory is defined as the path produced in the state space by the state vector **x(t)** as it changes with the passage of time. State space and state trajectory in the two-dimensional case are referred to as the phase plane and phase trajectory, respectively.

The first step in applying these definitions to a physical system is the selection of the system variables that are to represent the state of the system.

Note that there is no unique way of making this selection. The three common representations for expressing the system state are the physical, phase, and canonical state variables.

The selection of the state variables for the physical-variable method is based upon the energy-storage elements of the system. Table 1 lists some common energy-storage elements that exist in physical systems and the corresponding energy equations. The physical variable in the energy equation for each energy-storage element can be selected as a state variable of the system. Only independent physical variables are chosen to be state variables.

Independent state variables are those state variables that cannot be expressed in terms of the remaining assigned state variables. In some systems it may be necessary to identify more state variables than just the energy-storage variables. This situation is illustrated in some of the following examples, where velocity is a state variable. When position, the integral of this state variable, is of interest, it must also be assigned as a state variable.

For the circuit of Series RLC Circuit (Fig.2.2). This circuit contains two energy-storage elements, the inductor and capacitor. From Table 1, the two assigned state variables are identified as ***x1=Vc*** (the voltage across the capacitor) and ***x2=i*** (the current in the inductor). Thus two state equations are required.

Fig 2.3 is redrawn in Fig. 2.4 with node ***b*** as the reference node. The node equations for node ***a*** and the loop equations are, respectively,

$$C\dot{x} = x_2 \quad ; \quad L\dot{x}_2 + Rx_2 + x_1 = u$$

Rearranging terms to the standard state equation format yields:

$$\dot{x}_1 = \frac{1}{C} x_2$$

$$\dot{x}_2 = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u$$

Table 2.1.Energy storage elements

System	Elemnet	Energy	Physical variable
Electrical	Capacitor (C)	—	Voltage (v)
	Inductor (L)	—	Current (i)
Mechanical	Mass (M)	—	Translational velocity (v)
	Moment of inertia (J)	—	Rotational velocity (w)

	Spring (K)	—	Displacement (x)
Fulid	Fluid compressibility $\overline{(}$	—	Pressure (P_L)
	Fluid capacitor C=ρA	—	Height (h)
Thermal	Thermal capacitor C	—	Temperature (Θ)

Equation (2.13) represents the state equations of the system containing two independent state variables. Note that they are first-order linear differential equations and are ***n=2*** in number. They are the minimum number of state equations required to represent the system's future performance.

State equation. The state equations of a system are a set of **n** first-order differential equations, where **n** is the number of independent states.

The state equation represented by Eq(2.12)is expressed in matrix notation as:

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix} = \begin{bmatrix}0 & 1/C\\ -1/L & -R/L\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix}u \qquad (2.14)$$

The standard form of the state space equation is:

$$X^{\cdot} = Ax + Bu$$

In this case ,the matrix A which is the system matrix will be :

A= $\begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}$, nxn plant coefficient matrix

Matrix B ,which is input matrix will be :

B= $\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$, nx1 control matrix

and, in this case**,** u=[u] is a one-dimensional control vector.

In ($X^{\cdot} = Ax + Bu$), matrix **A** and **x** are conformable. If the output quantity y(t) for the circuit of Fig.8 is the voltage across the capacitor v_C, then

$$y(t) = v_c = x_1$$

Thus the matrix system output equation for this example is:

$$y(t) = Cx + Du = [1 \quad 0]\begin{bmatrix} x1 \\ x2 \end{bmatrix} + [0]u \qquad (2.15)$$

Where

C is the output matrix with 1xn dimension for single input single output system (SISO),

D is the forward matrix =0.

For a multiple-input multiple-output (MIMO) system, with m inputs and l outputs, these equations become:

$$\dot{X} = Ax + Bu \quad ; \quad y = Cx + Du \quad ;$$

Where: A = n × n plant or system matrix

B = n × m control or input matrix ; C = l × n output matrix

D = l × m feed forward matrix

u = m- dimensional control vector : y = l- dimensional output vector

The block diagram of the state space representation can be shown in the Fig.2.5.

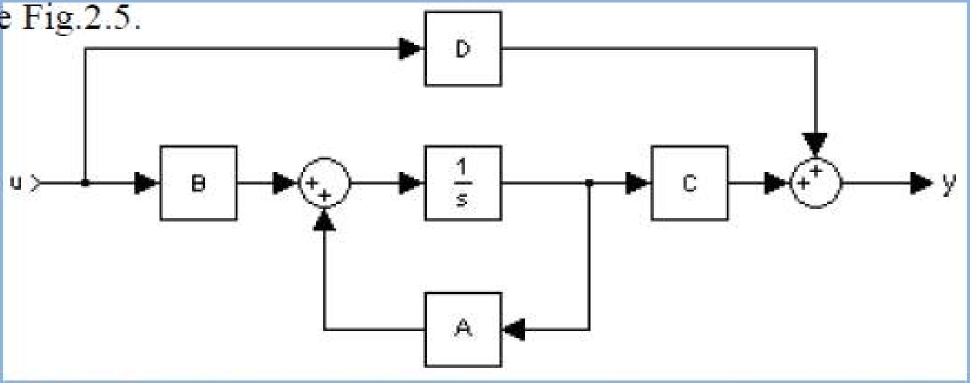


Fig.2.5. Block diagram of the linear, continuous time control system represented in state space.

2.5. Transfer Function (T.F):

If the system differential equation is linear, the ratio of the output variable to the input variable, where the variables are expressed as functions of the D operator, is called the transfer function.

Consider the system output **vC=y** in the RLC circuit of Fig.2.1 substituting **i=CDv$_C$** into Eq(LDi+Ri+1/CD i=e), yields:

$(LCD^2+RCD+1)v_c;(t)$

The system transfer function is:

$$G(D) = \frac{y(t)}{u(t)} = \frac{v_c(t)}{e(t)} = \frac{1}{LCD^2 + RCD + 1}$$

The notation G(D) is used to denote a transfer function when it is expressed in terms of the D operator. It may also be written simply as G.

The block diagram representation of this system (Fig.2.6) represents the mathematical operation G(D)u(t)=y(t); that is, the transfer function times the input is equal to the output of the block. The resulting equation is the differential equation of the system.

Fig.2.6. Block diagram representation.

Note: sometime used the symbol (s) instead of (D) and transfer function becomes writing as (G(s)) and D≡s and D2≡s2 ……, and the equation (2.16) will write as:

$$G = \frac{1}{LCS^2 + RCS + 1} \qquad (2.17)$$

The program by using Matlab to change between the two forms for representation of control system (State Space and Transfer Function) can be shown below;

% The change in form from SS to TF is:

```
Cy=1;Ly=10;Ry=100;

A=[ 0  /Cy  -1/Ly -Ry/Ly ];
```

```
B=[0  1/Ly];

C=[1 0];

D=[0];

[num,den]=ss2tf(A,B,C,D)

% The change in form from TF to SS is:

numc=[0 0 1];
denc=[Ly*Cy Ry*Cy 1];

[AA,BB,CC,DD]=tf2ss(numc,denc)
```

In the Matlab/Simulink can be done as in the Fig. 2.7.

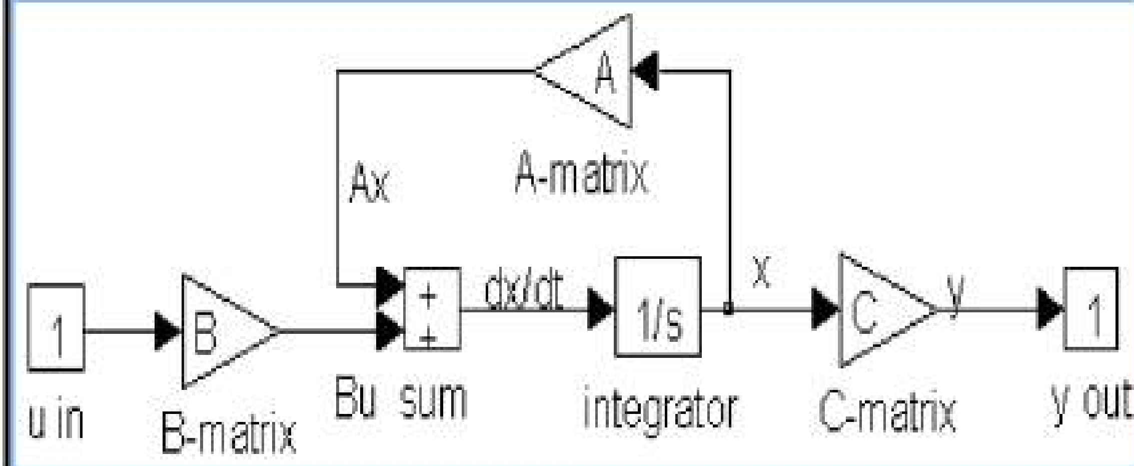


Fig.2.7. Simulink state space representation of control system

***H.W.* Implement the program and compare between two results?**

2.6. Correlation between transfer functions and state-space equations

The following full derivation of transfer function of SISO system from the state-space equations. Let us consider the system whose transfer function is given by:

$$G(s) = \frac{Y(s)}{U(s)}$$

This system may be represented in state space by the following equations:

$$X^{\cdot} = Ax + Bu \qquad ; \qquad y = Cx + Du$$

Where : x is the state vector ,u is the input and y is the output. The Laplace transform of the equation 2 is given by:

$$\left.\begin{aligned} sx(s) - x(0) &= Ax(s) + Bu(s) \\ Y(s) &= Cx(s) + Du(s) \end{aligned}\right\}$$

(2.20) Since the transfer function is previously defined as Laplace transformation of the output to the input with zero initial conditions, we assume that x(0)=0,then we have

$$sx(s) - Ax(s) = Bu(s) \text{ or } (SI - A)X(s) = BU(s)$$

Multiplying $(SI - A)^{-1}$ to both sides of the last equation we will obtain

$$X(s) = (SI - A)^{-1} BU(S)$$

Substitute (2.21) in (2.20) we get

$$Y(s) = [C(SI - A)^{-1} B + D]U(s)$$

So that the transfer function of the system represented by state space will be:

$$G(s) = Y(s)/U(s) = [C(SI - A)^{-1} B + D]$$

The right hand side of equation (2.23) involves $(SI - A)^{-1}$.Hence G(s) can be written as

$$G(s) = Q(s)/|(SI - A)|$$

Where Q(s) is a polynomial in s .Therefore, $|(SI - A)|$ is equal to the characteristic polynomial of G(s).in other words ,the Eigen values of A are identical to the poles of G(s).

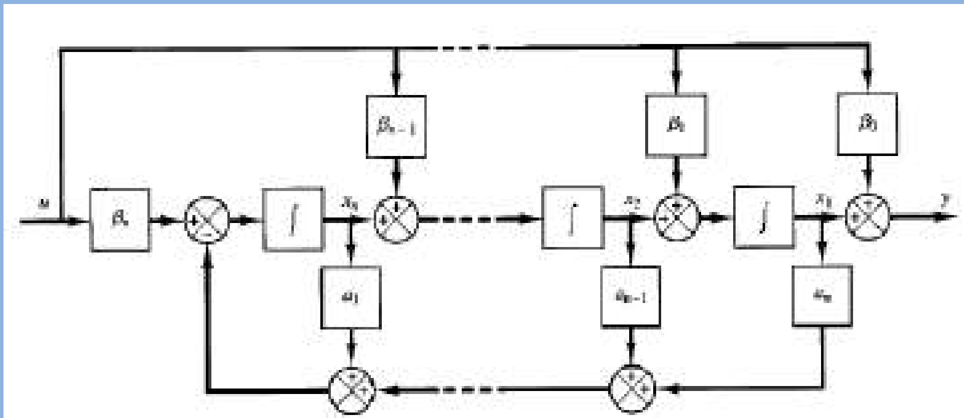


Fig.2.8. Block diagram of state equation and output equation

From Fig.2.8. can be written the following states equations:

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} = \dot{x}_2 - \beta_2 u$$

$$x_n = y^{n-1} - \beta_0 u^{n-1} - \beta_1 u^{n-2} - \dots\dots\dots\dots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

Where $\beta_0, \beta_1, \beta_2, \beta_n$ are determined from

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

.

.

$$\beta_n = b_n - a_1\beta_{n-1} - a_{n-2}\beta_{n-2} - \ldots\ldots - a_n\beta_0$$

With this choice of state variables, the existence and uniqueness of the solution of the state equation is guaranteed. (Note that this is not the only choice of a set of state variables). With the present choice of state variable, we obtain

$$\dot{x}_1 = x_2 + \beta_1 u$$

$$\dot{x}_2 = x_3 + \beta_2 u$$

$$\dot{x}_{n-1} = x_n + \beta_{n-1} u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \ldots\ldots - a_1 x_n + \beta_n u$$

In term of vector matrix equations, the output equation can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$[y] = [1 \quad 0 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} + \beta_0 u$$

Or $\dot{X} = AX + BU$ $Y = CX + DU$

Where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_{n-1} \\ x_n \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_1 \end{bmatrix}, B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$D = \beta_0 = b_0$$

Note that the state space representation for the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_n}$$

Example (1): Obtain the state equations for the circuit of Fig.2.9. The output is the voltage v_1.The input or control variable is a current source i(t). The assigned state variables

are i_1 , i_2 , i_3 , v_1 , and v_2 , ?

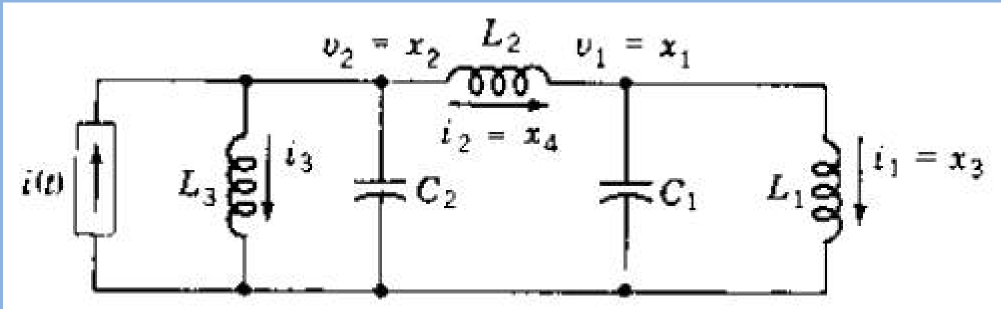


Fig.2.9. Circuit of Example 4.

Solution:

Three loop equations and two node equations are written:

$v_1 = L_1 D i_1$ or $v_1 = L_1 \frac{di_1}{dt}$

$$v_2 = L_2 D i_2 + v_1$$

$$v_2 = L_3 D i_3 \quad ; \quad i_2 = C_1 D v_1 + i_1$$

$$i = i_3 + C_2 D v_2 + i_2 \quad ; \quad L_3 i_3 = L_2 i_2 + L_1 i_1 + K$$

$$X^{\cdot} = \begin{bmatrix} 0 & 0 & -1/C_1 & -1/C_1 \\ 0 & 0 & -L_1/C_2 L_3 & -(L_2 + L_3)/C_2 L_3 \\ 1/L_1 & 0 & 0 & 0 \\ -1/L_2 & 1/L_2 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1/C_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$[y] = [1 \quad 0 \quad 0 \quad 0] X$$

2.7. Transfer Function from State-Variable Representation:

Having established the conditions for the equivalence of the state-variable representation with that of the transfer-function, we are interested to find one representation from the other by finding their relationship. Let us consider first the problem of determining the transfer function of a system given the state variable representation

$$X^{\cdot}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

since the transfer-function representation is expressed in the frequency domain, we begin by taking the Laplace transform of both equations, assuming as usual in transfer-function determination that the initial conditions on x are all zero.

$$SX = AX(s) + BU(s) \quad ; \quad Y(s) = CX(s)$$

Grouping the two X(s) terms in Equation (2.24) we have

$$(sI - A)X(s) = BU(s)$$

where the identity matrix has been introduced to allow the indicated multiplication compatible. Now, pre-multiplying both sides of the above equation by $(sI - A)^{-1}$, we get

$$X(s) = (sI - A)^{-1} BU(s)$$

We substitute this result in Equation (2.25) to obtain

$$Y(s) = C(sI - A)^{-1} BU(s)$$

Comparing this relation between Y(s) and U(s) with the Equation

$$(Y(s) = G(s)U(s))$$

we find that the transfer function matrix G(s) as:

$$G(s) = C(sI - A)^{-1} B$$

For the single input-single output case, this result reduces to

$$G(s) = c'(sI - A)^{-1} b$$

The matrix $(sI - A)^{-1}$ is commonly referred to as the resolving matrix and is designated by $\varphi(s)$,

$$\varphi(s) = (sI - A)^{-1}$$

In terms of this notation the Equations become

$$G(s) = C\,\varphi(s)\,B \text{ and } G(s) = c'\,\varphi(s)\,b \qquad (2.26)$$

Example (2): Consider the system represented by the equations

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

The matrix $(sI - A)$ in this example becomes

$$s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 10 & s+7 \end{bmatrix}$$

Its inverse is found as

$$\phi(s) = (sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)} = \frac{\begin{bmatrix} s+7 & 1 \\ -10 & s \end{bmatrix}}{s^2 + 7s + 10}$$

Hence, transfer function will be as

$$G(s) = c'\phi(s)b = (sI - A)^{-1} = \frac{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} s+7 & 1 \\ -10 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 7s + 10} = \frac{1}{s^2 + 7s + 10}$$

In the above example, we observe that the determinant of the matrix (sI – A) is equal to the denominator polynomial of G(s). This is always true for single input - single output systems. Although Equation (2.26) provide a direct method for determining the transfer function of a system from a state-variable representation of the system, it is generally not the most efficient method.

2.8. State Variable Representation from Transfer Function:

In Section 2.6 we have shown how to get the transfer function model of a linear continuous system when its state-variable form is available. We shall now take up the issue of getting the state-variable model when the transfer function model is available. Since the state-variable representation is not unique, there are, theoretically, an infinite number of ways of writing the state equations. We shall present here one method for deriving a set of continuous state variable representation from the transfer function. Analogous procedure may be followed for writing the continuous state equation from pulse transfer function in S domain. The transfer function of single-input-single-output system of the form:

$$G(s)=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+.......+b_1s+b_0}$$

Can be written ,after introducing an auxiliary variable E(s) as

$$G(s)=\frac{Y(s)}{U(s)}=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+.......+b_1s+b_0}*\frac{E(s)}{E(s)}$$

We let now

$$Y(s)=(a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0)E(s)$$

$$U(s) = (s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots\ldots + b_1 s + b_0)E(s)$$

From Theorem of Laplace transform, we note the following relations between the variables in the *s* domain and time domain with zero initial conditions

$$E(s) \rightarrow e(t)$$

$$sE(s) \rightarrow \dot{e}(t)$$

$$s^2 E(s) \rightarrow \ddot{e}(t)$$

Under this correspondence we define the state variables

$$x_1(t) = e(t)$$

$$x_2(t) = x_1^{\cdot}(t) = e^{\cdot}(t)$$

$$x_3(t) = x_2^{\cdot}(t) = e^{\cdot\cdot}(t)$$

.

.

$$x_n(t) = x_{n-1}^{\cdot}(t) = e^{n-1}(t)$$

From above two Equations group we obtain the state equations

$$x^{\cdot}_1(t) = x_2(t)$$

$$x^{\cdot}_2(t) = x_3(t)$$

$$x^{\cdot}_3(t) = x_4(t)$$

.

$$\dot{x}_n(t) = x_n(t) = -b_0 x_1(t) - b_1 x_2(t) - b_2 x_3(t) - b_{n-1} x_n(t) + u$$

In matrix notation this becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -b_0 & -b_1 & -b_2 & -b_3 & -b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

In compact form, it is written as :

x’(t) = A x(t) + Bu(t) ; The output equation is obtained from as

which may be written compactly a s **y**(t) = C**x**(t)

Hence the last two Equations are a set of state equations for the continuous system described by transfer function. Another convenient and useful representation of the continuous system is the signal flow graph or the equivalent simulation diagram. These two forms can be derived, after dividing both the numerator and denominator of first Equation by *sn* :

$$G(s)=\frac{Y(s)}{U(s)}=\frac{a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+.......+b_1s+b_0}*\frac{E(s)}{E(s)}$$

From this expression we can get two equations

$$Y(s)=(a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+......+a_1s+a_0)E(s)$$

$$U(s) = (s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots\ldots + b_1 s + b_0)E(s)$$

The above equation can be rewritten as follows

$$E(s) = U(s) - b_{n-1}s^{-1}E(s) - b_{n-2}s^{-2}E(s) + \ldots\ldots - b_1 s^{1-n}E(s) - b_0 s^{-n}E(s)$$

Example (3): Let us consider a single input single output system of the last Example which is reproduced below for quick reference:

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

We are interested to find its solution with initial condition $x'(t_0) = x'(0) = [0\ 0]$ and unity step input $u(t) = us(t)$. The resolving matrix $\varphi(s)$ given by relation ($\varphi(s) = (s\mathrm{I} - \mathrm{A})^{-1}$) is written as :

$$\phi(s) = (sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)} = \frac{\begin{bmatrix} s+7 & 1 \\ -10 & s \end{bmatrix}}{s^2 + 7s + 10}$$

$$\phi(s) = \begin{bmatrix} \dfrac{s+7}{s^2+7s+10} & \dfrac{1}{s^2+7s+10} \\ \dfrac{-10}{s^2+7s+10} & \dfrac{s}{s^2+7s+10} \end{bmatrix}$$

$$\phi(s)=\begin{bmatrix}\frac{1}{3}\left(\frac{5}{s+2}-\frac{2}{s+5}\right) & \frac{1}{3}\left(\frac{1}{s+2}-\frac{1}{s+5}\right)\\ \frac{10}{3}\left(\frac{1}{s+2}-\frac{1}{s+5}\right) & \frac{1}{3}\left(\frac{5}{s+5}-\frac{2}{s+25}\right)\end{bmatrix}$$

$$\phi(t)=\begin{bmatrix}\frac{1}{3}(5e^{-2}-2e^{-5}) & \frac{1}{3}(1e^{-2}-1e^{-5})\\ \frac{10}{3}(1e^{-2}-1e^{-5}) & \frac{1}{3}(5e^{-5}-2e^{-2})\end{bmatrix}$$

Substituting the value of x'(0)=[0 0] and unit step input in the equation we get

$$x(t)=\int_0^t \phi(t-\tau)bu(\tau)d\tau=\begin{bmatrix} \frac{1}{3}\int_0^t (5e^{-2(t-\tau)}-2e^{-5(t-\tau)})d\tau & \frac{1}{3}\int_0^t (1e^{-2(t-\tau)}-1e^{-5(t-\tau)})d\tau \\ \frac{10}{3}\int_0^t (1e^{-2(t-\tau)}-1e^{-5(t-\tau)})d\tau & \frac{1}{3}\int_0^t (5e^{-5(t-\tau)}-2e^{-2(t-\tau)})d\tau \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(t)=\begin{bmatrix} \frac{1}{3}\int_0^t (1e^{-2(t-\tau)}-1e^{-5(t-\tau)})d\tau \\ \frac{10}{3}\int_0^t \quad \frac{1}{3}\int_0^t (5e^{-5(t-\tau)}-2e^{-2(t-\tau)})d\tau \end{bmatrix}=\begin{bmatrix} \frac{1}{10}-\frac{1}{6}e^{-2t}+\frac{1}{15}e^{-5t} \\ \frac{1}{3}e^{-2t}-\frac{1}{3}e^{-5t} \end{bmatrix}$$

Therefore y(t) is computed as *y(t)=cx(t)+du(t)*

$$y(t)=\frac{1}{10}-\frac{1}{6}e^{-2t}+\frac{1}{15}e^{-5t},t>0$$

9. Properties of the State Transition Matrix:

Some useful properties of the state transition matrix $\varphi(t)$ are recorded below :

1. $\phi(0) = e^{A0} = I$ (identity matrix)

2. $\phi(t) = e^{At} = e^{(-At)^{-1}} = \phi(-t)^{-1}$ or $\phi(-t)^{-1} = \phi^{-1}(t) = \phi(-t)$

3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}.e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

4. $(\phi(t))^n = \phi(nt)$

5. $\phi(t_1 - t_2)\phi(t_2 - t_3) = \phi(t_1 - t_3)$ for any t_1, t_2, t_3

6. $\frac{\phi(t)}{dt} = A\phi(t)$

2.10. Complex impedances.

Consider the circuit shown in Fig.2.10. then the T.F of this circuit is

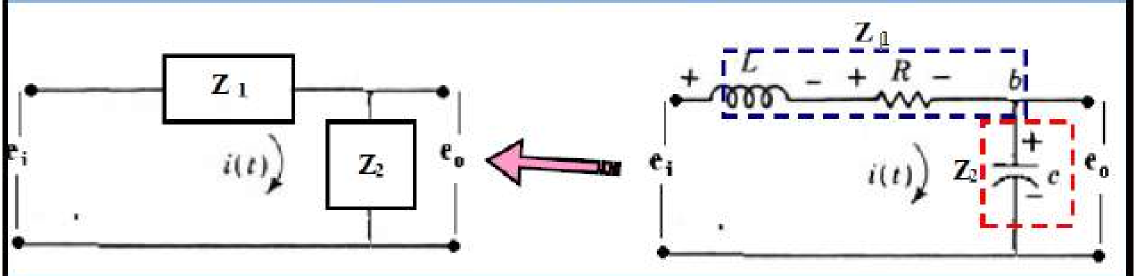


Fig.2.10. circuit diagram of complex impedance

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

Where

$$Z_1(s) = Ls + R \quad ; \quad Z_2(s) = \frac{1}{Cs}$$

$$Z(s) = E(s)/I$$

Hence the T.F. Eo(s)/Ei(s) can be found as follows;

$$\frac{E_0(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

Example (4): Consider the electrical cct shown in Fig.2.11. Obtain the T.F Eo(s)/Ei(s) by use of the block diagram approach?

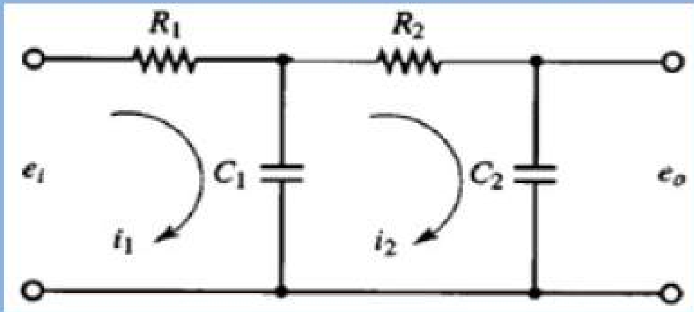


Fig. 2.11. Circuit diagram of Example 5.

Solution:

Equations of the circuit in Fig.2.11 are:

$$e_i = \frac{1}{C_1}\int (i_1 - i_2)dt + i_1 R_1$$

$$0 = \frac{1}{C_1}\int (i_2 - i_1)dt + \frac{1}{C_2}\int i_2 dt + i_2 R_2$$

$$e_0 = \frac{1}{C_2}\int i_2 dt$$

By taking the L.T. for the above equation:

$$E(s)_i = \frac{1}{C_1 s}(I_1(s) - I_2(s) + I_1(s)R_1$$

(2.27)

$$0=\frac{1}{C_1 s}(I_2(s)-I_1(s)+\frac{1}{C_2 s}I_2(s)I_2(s)R_2 \qquad (2.28)$$

$$E_0(s)=\frac{1}{C_2 s}I_2(s)$$

By using Eq(2.27), we get:

$$C_1 s[E(s)_i-I_1(s)R_1]=I_1(s)-I_2(s) \qquad (2.29)$$

From Eq's(2.28)&(2.29) we get:

$$I_2(s)=\frac{C_2 s}{R_2 C_2 s+1}*\frac{1}{C_1 s}[I_1(s)-I_2(s)]$$

The transfer function of Eo(s)/Ei(s) can written in term of I2(s) as follow:

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{(R_1C_1s+1)(R_2C_2s+1)+R_1C_2s} \qquad (2.30)$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits .since $(R_1C_1 + R_2C_2 + R_1C)^2 > (4R_1C_1R_2C_2)$ the two roots of the denominator of equation (2.30) are real. The present analysis show that if two RC circuits are connected cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1 C_1 s+1)$ and $1/(R_2 C_2 s+1)$.

The reason for this is that when we derive the transfer function for an isolated circuit, we implicitly assume to be infinite which means that no power is being withdrawn at the output. when the second circuit is connected to the output of the first , however, a certain amount of power is with –drawn and thus assumption of no loading then violated therefore if the transfer function of this system is obtained under the assumption of no loading then it is not valid . The degree of the loading effect determines the amount of modification of the transfer function.

2.11. Transfer functions of non-loading cascaded elements

The transfer function of a system consisting of two no loading cascaded elements can be obtained by eliminating the intermediate input and output. For example consider the system shown in Fig.2.12.a. The transfer functions of the elements are:

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \quad \text{and} \quad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

If the input impedance of the second elements is infinite, the input of the first element is not affected by connecting it to second element. Then transfer function of whole system becomes:

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)\ X_3(s)}{X_1(s)\ X_2(s)}$$

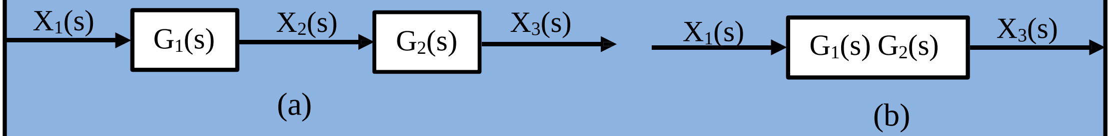
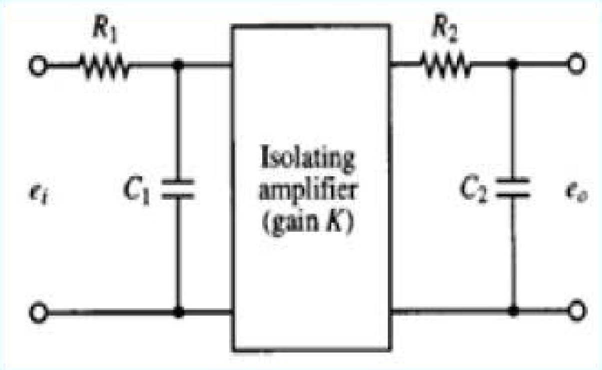


Fig. 2.12. (a) System consisting of two non-loading cascaded elements; (b) an equivalent system.

The transfer function of whole system is thus the product of transfer functions of the individual elements. This is shown in Fig.2.12.b.as an example, consider the system shown in Fig.2.13,the insertion of an isolating amplifier between the circuits to obtain non-loading characteristics is frequently used in combining circuits. Since the amplifier has very high input impedance, an isolation amplifier

inserted between two circuits justifies the non-loading assumption. The two simple RC circuit, isolated by an amplifier as shown in Fig.2.13. Have negligible effects and the transfer function of the entire circuit is equal to product of the individual transfer functions, thus in this case:
$$\frac{E_0(s)}{E_i(s)} = (\frac{1}{R_1C_1s+1})K(\frac{1}{R_2C_2s+1})$$

$$\frac{E_0(s)}{E_i(s)} = (\frac{K}{(R_1C_1s+1)(R_2C_2s+1)}$$



Example (5): Armature-Controlled dc motors

The dc motors have separately excited fields. They are either armature controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals. Consider the armature-controlled dc motor shown in the following Fig.2.14.

R_a = armature-winding resistance, ohms

L_a = armature-winding inductance, henrys

i_a = armature-winding current, amperes

i_f = field current, a-pares ; ea = applied armature voltage, volt

e_b = back emf, volts; θ = angular displacement of the motor shaft, radians

T = torque delivered by the motor, Newton*meter

J = equivalent moment of inertia of the motor and load referred to the motor shaft kg.m2

f = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft. Newton*m/rad/s

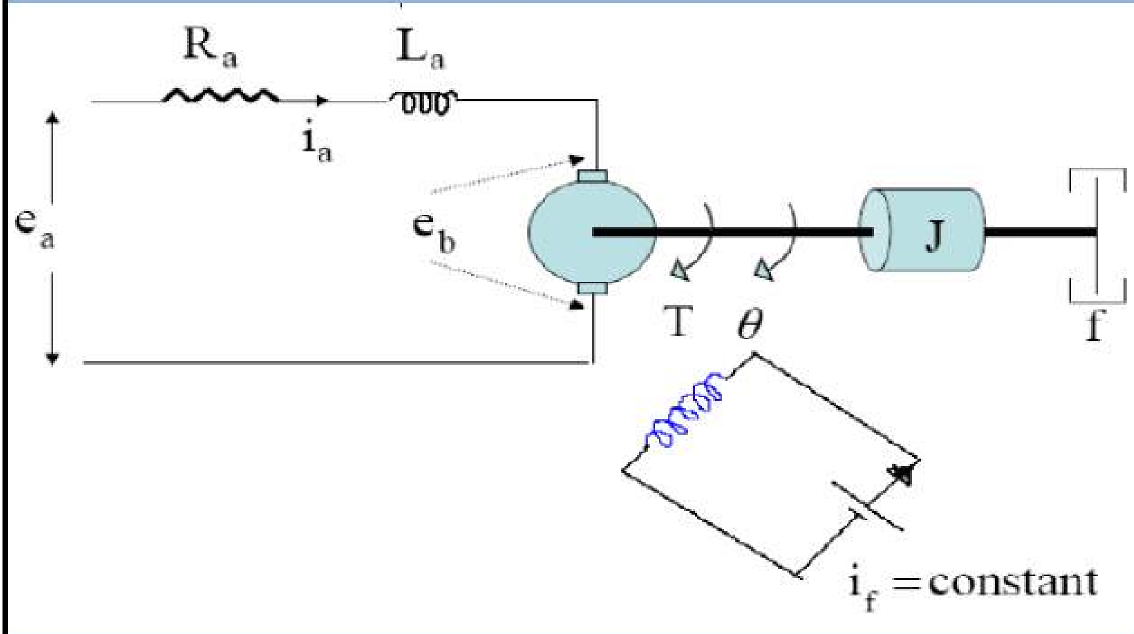


Fig.2.14. Circuit Diagram of armature-controlled dc motor

$T = K_1 i_a \psi$; Where

ψ is the air gap flux, $\psi = K_f i_f$,k_f is constant

$T = K_1 i_a K_f i_f$

For a constant field current

$T = K i_a$,k is a motor torque constant

$e_b = k_2 \psi \omega$

For constant flux

$e_b = k_b \frac{d\vartheta}{dt}$, k_b is back emf constant ; The armature circuit equation is

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a$$

The armature current produces torque which is applied to the both inertia and friction to rotate the motor, hence

$$J \frac{d^2\vartheta}{dt} + f \frac{d\vartheta}{dt} = T = k i_a$$

Taking the Laplace transform of the above three equation with assuming all initial condition is zero

$$E_b(s) = K_b s\theta(s)$$

$$L_a s I_a(s) + R_a I_a(s) + E_b(s) = E_a(s)$$

$$(L_a s + R_a) I_a(s) + E_b(s) = E_a(s)$$

$$Js^2\theta + fs\theta = T = KI_a(s)$$

$$(Js^2 + fs)\theta = T = KI_a(s)$$

The transfer function can be obtained as:

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s(L_a Js^2 + (L_a f + R_a J)s + R_a f + KK_b}$$

Try to check this equation

Example (6): Field-Controlled dc motor

Find the T.F (**Θ(s)/Ef(s))**For the field-controlled dc motor shown in Fig.2.15 below

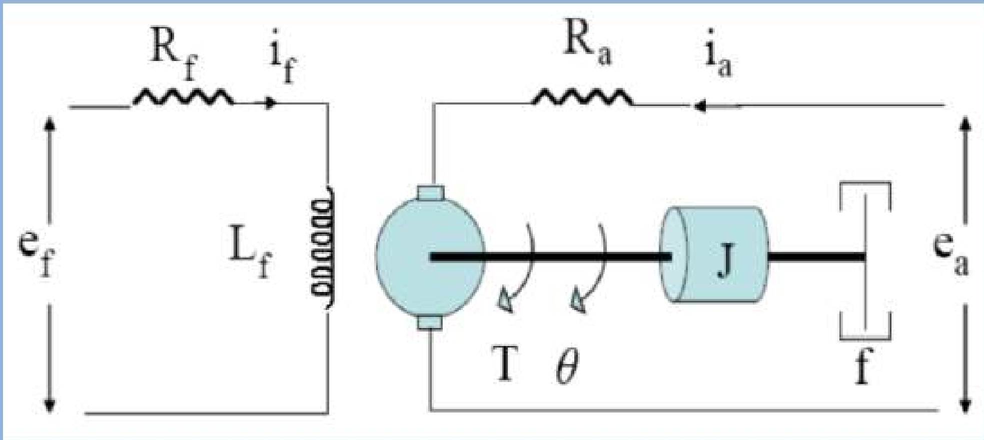


Fig.2.15. Circuit Diagram of field-controlled dc motor

The torque T developed by the motor is proportional to the product of the air gap flux ψ and armature current i_a so that

$T = K_1 i_a \psi$, k_1 is constant

$T = K_2 i_f$, k_2 is constant

$$L_f \frac{di_f}{dt} + R_f i_f = e_f \qquad ;$$

$$J \frac{d^2\vartheta}{dt} + f \frac{d\vartheta}{dt} = T = k_f i_f$$

Taking the Laplace transform of the above three equation with assuming all initial condition is zero

$$(L_f s + R_f) I_f(s) = E_f(s)$$

$$(Js^2 + fs)\theta = T = K_2 I_f(s)$$

The transfer function can be obtained as

$$\frac{\theta(s)}{E_a(s)} = \frac{K_2}{s(L_f s + R_f)(Js + f)}$$

Try to check this equation

H.W. Find the transfer function $\frac{\theta(s)}{I_f(s)}$ and $\frac{E_f(s)}{I_f(s)}$

2.12. Mechanical Systems:

Mechanical systems obey Newton's law that the sum of the forces equals zero; that is, the sum of the applied forces must be equal to the sum of the reactive forces. The three qualities characterizing elements in a mechanical translation* system are mass, elastic, and damping. The following analysis includes only linear functions. Static friction, Coulomb friction, and other nonlinear friction terms are not included. Basic elements entailing these qualities are represented as network elements, and a mechanical network is drawn for each mechanical system to facilitate writing the differential equations. The mass M is the inertial element. A force applied to a mass produces an acceleration of the mass. The reaction force ***fM*** is equal to the product of mass and acceleration and is opposite in direction to the applied force. In terms of displacement ***x***, velocity ***v***, and acceleration ***a***, the force equation is

$$f_m = M_a = MD_v = MD^2x$$

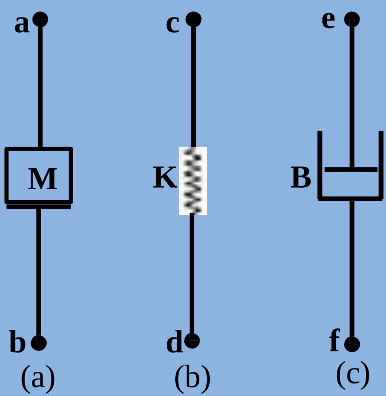


Fig. 2.16. Network elements of mechanical translate.

The network representation of mass is shown in Fig. 2.16.a.One terminal, ***a***, has the motion of the mass; and the other terminal, b, is considered to have the motion of the reference. The reaction force ***fM*** is a function of time and acts "through "*M*. The elastance, or stiffness, K provides a restoring force as represented by a spring. Thus, if stretched, the string tries to contract; if compressed, it tries to expand to its normal length. The reaction force ***fk*** on each end of the spring is the same and is equal to the product of the stiffness K and the amount of deformation of the spring. The network representation of a spring is shown in Fig.16b. The displacement of each end of the spring is measured from the original or equilibrium position. End c has a position xc, and end d has a position xd, measured from the respective equilibrium positions. The force equation, in accordance with Hooke's law, is

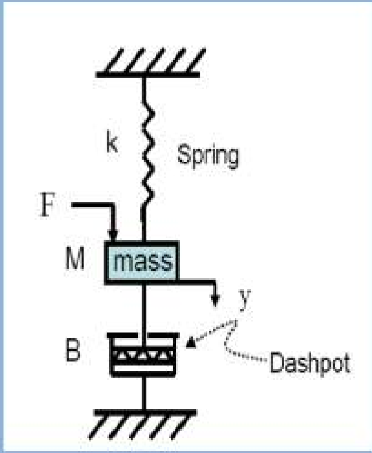
$$f_K = K(x_c - x_d)$$

If the end d is stationary, then xd =0 and the preceding equation reduces to: $f_K = Kx_c$

The plot ***fk*** vs. **xc** for a real spring is not usually a straight line, because the spring characteristic is nonlinear. However, over a limited region of operation, the linear approximation, i.e., a constant value for K, gives satisfactory results.

2.12.1. Translational mechanical system:

Example (7): find the transfer function of the following system



$$\sum F = ma$$

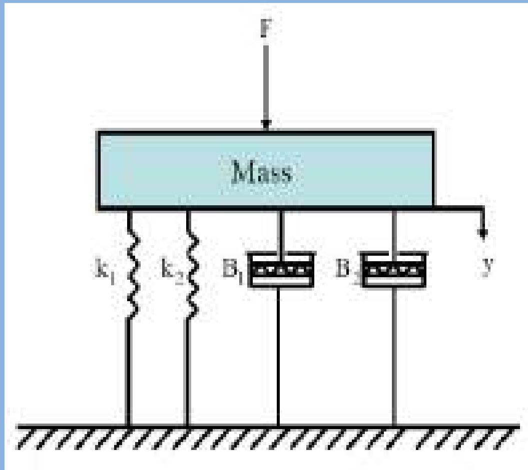
$F = Ky + mD^2y + BDy$,K is spring constant.

B viscous friction coefficient

$$F = mD^2y + BDy + ky$$

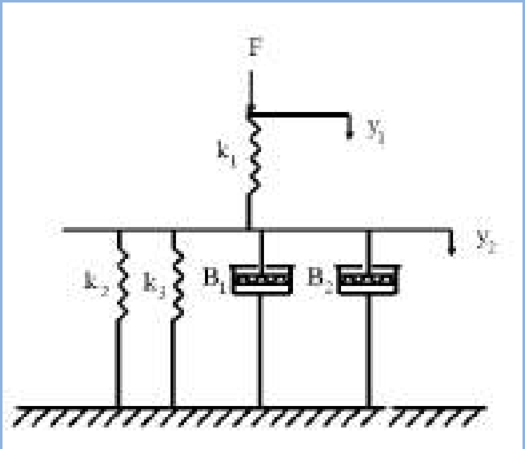
$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + Bs + k}$$

Example (8): Find the transfer function of the following system



$$F = mD^2y + (B_1 + B_2)Dy + (k_1 + k_2)y \; ; \quad \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + (B_1 + B_2)s + (k_1 + k_2)}$$

Example (9): Find the transfer function of the following system

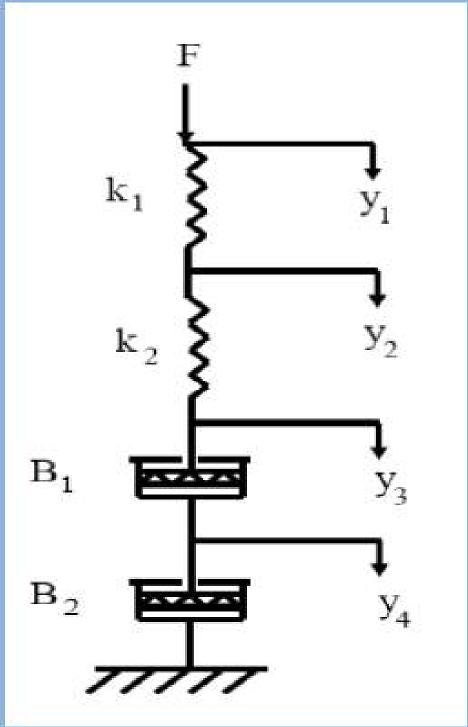


$$F = k_1(y_1 - y_2)$$

$$k_1(y_1 - y_2) = (k_2 + k_3)y_2 + (B_1 + B_2)Dy_2$$

Example (10): Find the transfer function of the following system

$$F = k_1(y_1 - y_2) = k_2(y_2 - y_3)$$
$$= B_1 D(y_3 - y_4) = B_2 D y_4$$



2.12.2. Rotational mechanical systems:

$$J\alpha = \sum T$$

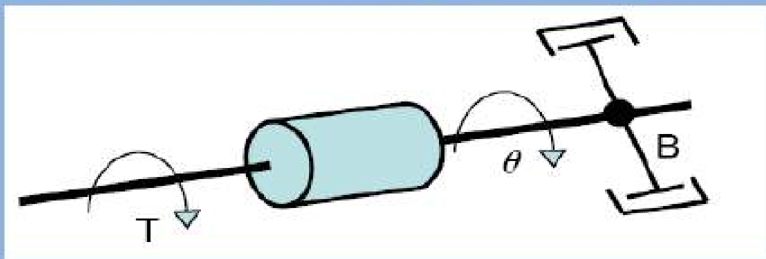
Where

J is the moment of inertia

α is the rotational acceleration

T is the toque

Example(11): Write the mathematical model of the following system



$$T = JD^2\theta + BD\theta$$

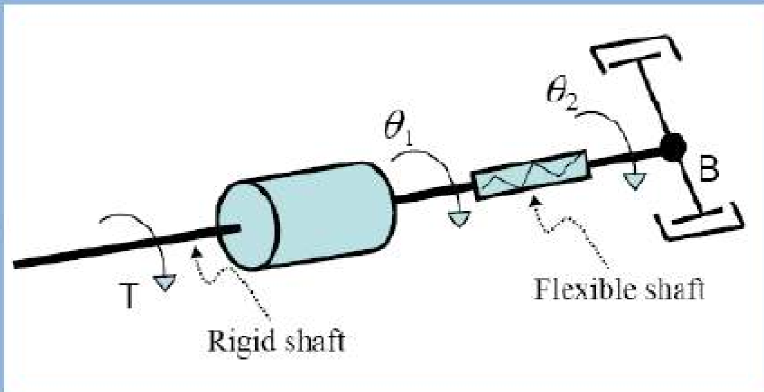
$$T = JD\omega + B\omega$$

Where

$\omega = \dot{\theta} = D\theta$

$\alpha = \dot{\omega} = \ddot{\theta} = D^2\theta$

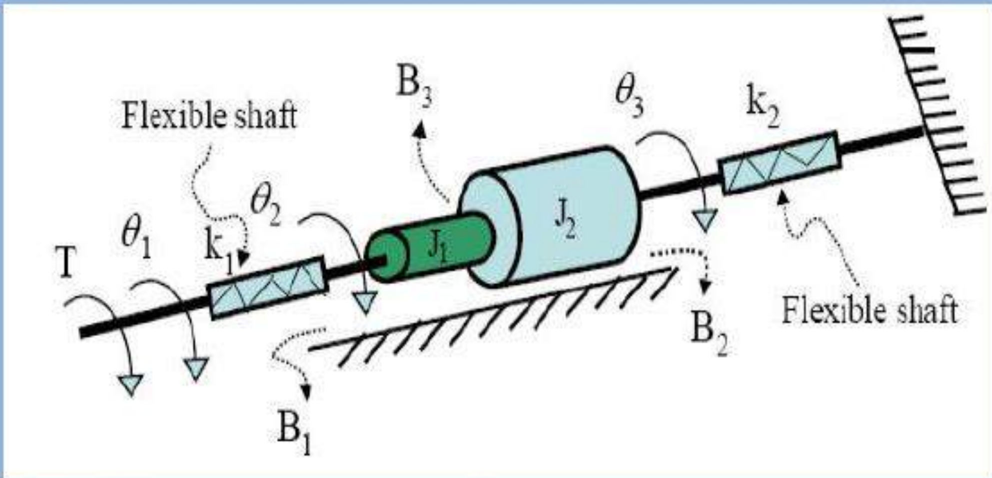
Example (12): Write the mathematical model of the following system



$T = JD^2\theta_1 + K(\theta_1 - \theta_2)$

$K(\theta_1 - \theta_2) = BD\theta_2$

Example (13): Write the mathematical model of the following system

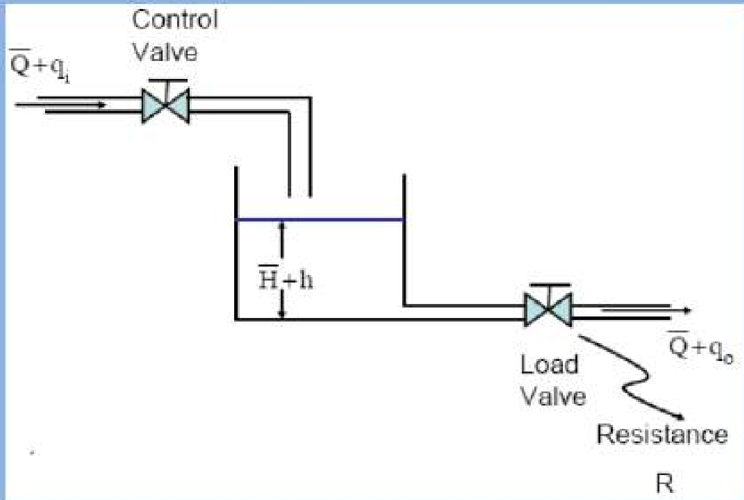


$T = K_1(\theta_1 - \theta_2)$; $T = K_1(\theta_1 - \theta_2) = J_1 D^2\theta + B_3 D(\theta_2 - \theta_3) + B_1 D\theta_2$

$B_3 D(\theta_2 - \theta_3) = J_2 D^2\theta_3 + B_2 D\theta_3 + K_2\theta_3$

2.13. Liquid level systems

Example1.Write the mathematical model of the following system



$$q_i - q_0 = c\frac{dh}{dt}$$

Where

For laminar flow

$$q_0 = \frac{h}{R}$$

For turbulent flow

$$q_0 = K\sqrt{R}$$

$$q_i - \frac{h}{R} = c\frac{dh}{dt}$$

The differential equation will be

$$Rc\frac{dh}{dt} + h = Rq_i$$

Taking the Laplace transform

$$(Rcs + 1)H(s) = RQ_i(s)$$

So that the final transfer function will be

$$\frac{H_0(s)}{Q_i(s)} = \frac{R}{(Rcs + 1)}$$

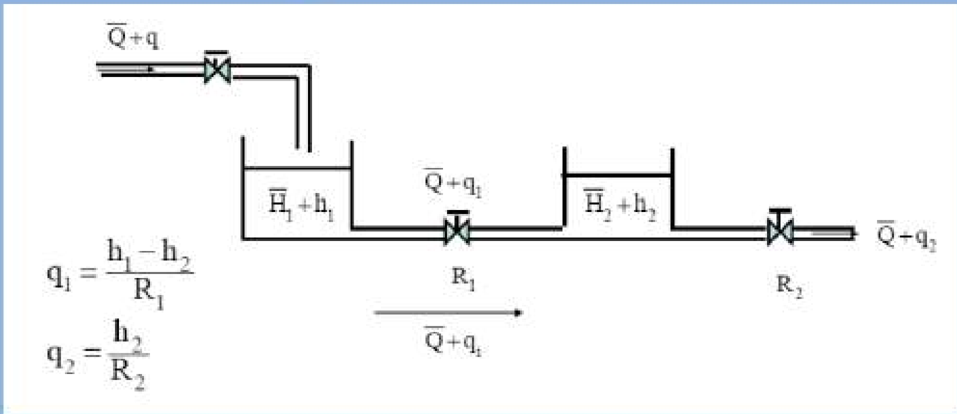
According to that we can find

$$\frac{Q_0(s)}{Q_i(s)} = \frac{1}{(Rcs + 1)}$$

Example (14): Find the TF of the interaction liquid level system

$$q_1 = \frac{h_1 - h_2}{R_1} \;:\; q_2 = \frac{h_2}{R_2} \;:$$

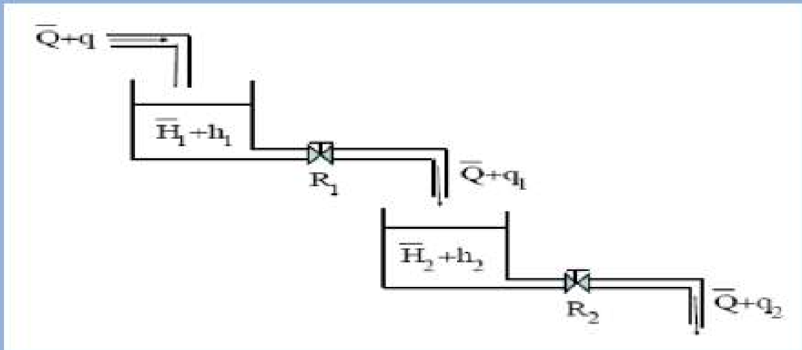
$$C_1 \frac{dh_1}{dt} = q - q_1 \;:\; C_2 \frac{dh_2}{dt} = q_1 - q_2 \;:$$



$$\frac{Q_2(s)}{Q(s)} = \frac{1}{R_1c_1R_2c_2s^2 + s(R_1c_1 + R_2c_2 + R_2c_1) + 1}$$

Home work: Find TFs. $\frac{Q_1(s)}{Q(s)}, \frac{H_1(s)}{Q(s)}, \frac{H_2(s)}{Q(s)}$

Example (15). Find the TF of the non-interaction liquid level system

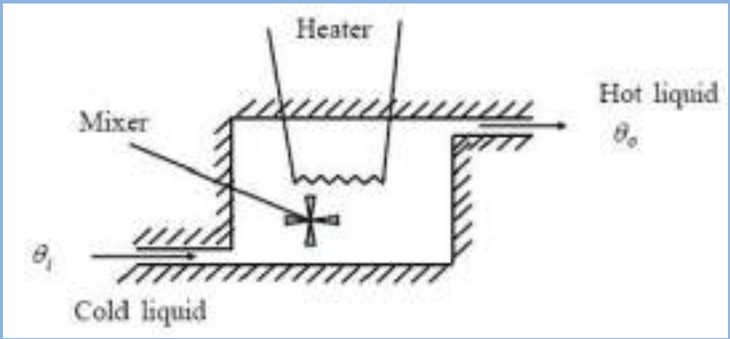


$$q_1 = \frac{h_1}{R_1}$$

$$q_2 = \frac{h_2}{R_2} \quad ; \quad C_1 \frac{dh_1}{dt} = q - q_1 \quad ; \quad C_2 \frac{dh_2}{dt} = q_1 - q_2$$

Home work : Find TFs. $\frac{Q_2(s)}{Q(s)}$

2.14. Thermal Systems:



Consider that heat input rate changes from $\overline{H}$ to $\overline{H}+h_i$ then heat outflow will change from $\overline{H}$ to $\overline{H}+h_o$ also the temperature of the out following liquid will change from $\overline{\theta_o}$ to $\overline{\theta_o}+\theta_o$. Considering change only:

$$h_i - h_o = Q\frac{d\theta}{dt}$$

$$\theta = h.R$$

Or

$$RQ\frac{d\theta}{dt} + \theta = Rh_i$$

Note

$$h_o = Gc\theta \quad ; \quad Gc = \frac{1}{R} \quad ; \quad Q = Mc$$

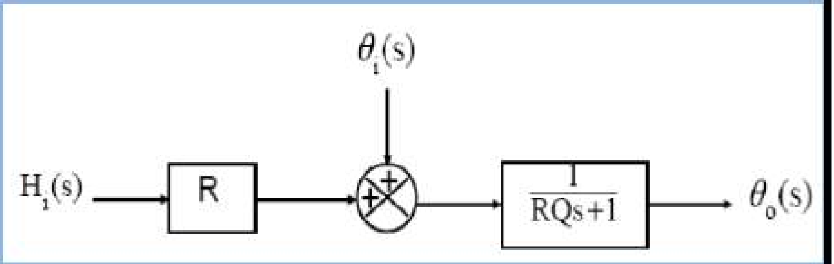
By taking Laplace transform

$$\frac{\theta(s)}{H_i(s)} = \frac{R}{RQs+1}$$

$$Q\frac{d\theta}{dt} = Gc\theta_i - h_o$$

$$Q\frac{d\theta}{dt} = \frac{1}{R}\theta_i - \frac{\theta}{R}$$

$$RQ\frac{d\theta}{dt} + \theta = \theta_i$$



$$\frac{\theta(s)}{\theta_i(s)} = \frac{1}{RQs+1}$$

In case of changes in both θ_i & h_i then we have :

$$Rc\frac{d\theta}{dt} + \theta = \theta_i + Rh_i$$

Where

θ_i : Steady state temperature of inflowing liquid, F^o.

θ_o : Steady state temperature of outflowing liquid, F^o.

G : steady state liquid flow rate lb/sec.

M: mass of liquid in tank,lb.

c: specific heat of liquid tu/lb.F^o.

R: thermal resistance, Fo sec/B tu.

Q: thermal capacitance, B tu/F^o.

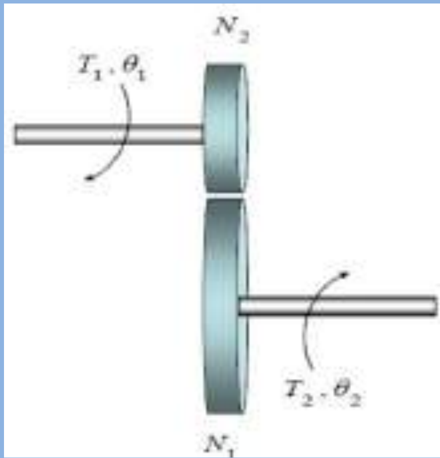
$\overline{H}$: Steady state heat i/p rate ,B tu/sec.

15. Extra systems

1. ear trains

A gear train is a mechanical device that transmit energy from one part of a system to another in such a way that force, torque ,speed and displacement are altered. Two gears are shown coupled together in following figure. The inertial and friction of the gears are neglected in the ideal case considered.

The relationships between the torque T_1, T_2 and angular displacements θ_1, θ_2 and the teeth numbers N_1, N_2 of the gear train are derived from the following facts.



1. The number of teeth on the surface of the gear is proportional to the radius r_1, r_2 of the gears ,that is.

$$r_1 N_2 = r_2 N_1$$

2. The distance traveled along the surface of each gear is same. Therefore

$$r_1\ \theta_1 = r_2\ \theta_2$$

3. The work done by one gear is equal to that of the other since there is assume to be no loss, thus

$$T_1\ \theta_1 = T_2\ \theta_2$$

If the angular velocities of the two gears are ω_1 and ω_2

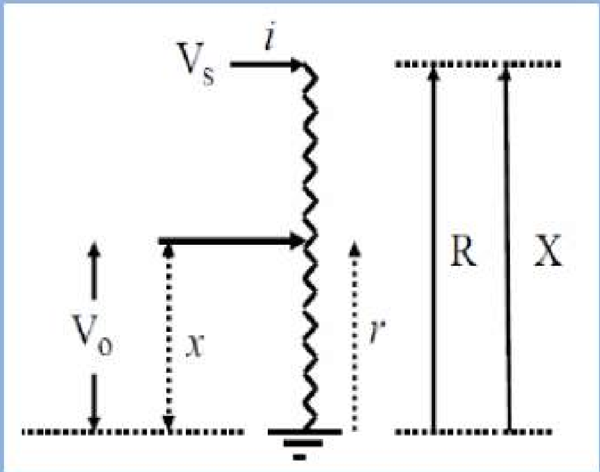
$$\frac{T_1}{r_1\, T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{}{r_2}$$

2.15.2. Potentiometer (transducer):

Consider linear resistance

$x = kr \quad ; \quad X = kR$

$i = \dfrac{V_s}{R} \quad ; \quad V_o = iR$



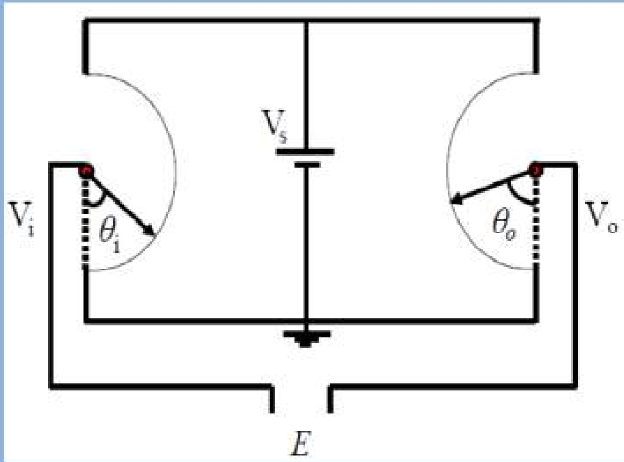
From 1 and 2

$V_o = \dfrac{V_s}{R} r$

$V_o = \dfrac{V_s}{\frac{X}{k}} \cdot \dfrac{x}{k} = \dfrac{V_s}{X} x$

If $K_p = \dfrac{V_s}{X} \quad ; \quad V_o = K_p x$

2.15.3. Error Detector



$V_i = k_1\theta_1$

$V_o = k_2\theta_2$

$E = k_1\theta_1 - k_2\theta_2$

If $k_1 = k_2 = k$

$E = k\ (\theta_1 - \theta_2)$

2.15.4. First-Order Op-Amp:

In addition to adding and subtracting signals, op-amps can be used to implement transfer functions of continuous-data systems. While many alternatives are available, we will explore only those that use the inverting op-amp configuration shown in beside Fig.

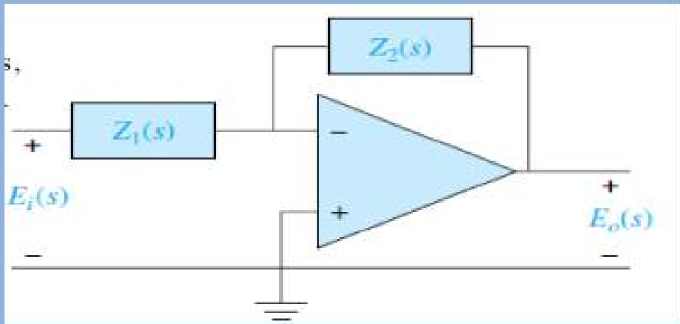


Fig.2.17. Inverting op-amp configuration

In the figure, Z1(s) and Z2(s) are impedances commonly composed of resistors and capacitors. Inductors are not commonly used because they tend to be bulkier and more expensive. Using ideal op-amp properties, the input-output relationship, or transfer function, of the circuit shown in Fig. can be written in a number of ways, such as

$$G(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{1}{Z_1(s)Y_2(s)} = -Z_2(s)Y_1(s) = -\frac{Y_1(s)}{Y_2(s)}$$

Where $Y1(s) = 1/Z1(s)$ and $Y2(s)=1/Z(s)$ are the admittances associated with the circuit impedances.

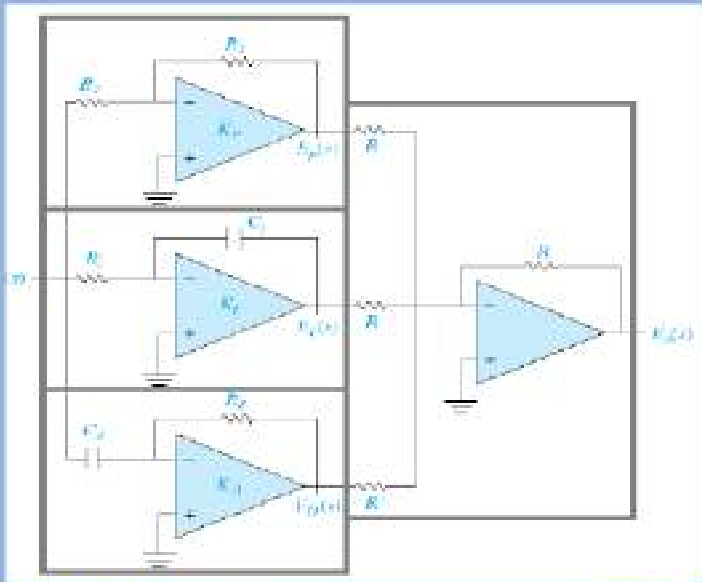
Example (16): For the Fig. below find The transfer function that Described the relation between Eo(s) and E(s) (i.e. Find **G(s)= Eo(s)/ E(s)?**

Solution:

$$\frac{E_p(s)}{E(s)} = -\frac{R_2(s)}{R_1(s)}$$

$$\frac{E_1(s)}{E(s)} = -\frac{1}{R_i c_i s}$$

$$\frac{E_D(s)}{E(s)} = -R_D c_D s$$



The output voltage is

$$E_o(s) = -[E_p(s) + E_1(s) + E_D(s)]$$

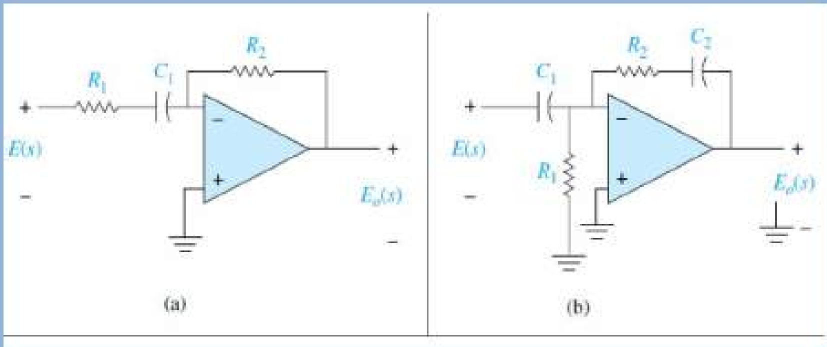
Thus the transfer function of PID operation amplifier is

$$G(s) = \frac{E_0(s)}{E(s)} = \frac{R_2}{R_1} + \frac{1}{R_i c_i s} + R_D c_D s$$

$$G(s) = \frac{R_1 R_i c_i R_d c_d s^2 + R_1 R_i c_i s + R_1}{R_1 R_i c_i s}$$

This is transfer function of (proportional , integral , derivative)(PID) controller that will study in details in next time.

H.W. Find the transfer functions Eo(s)/E(s) for each the circuits shown in (a, b) of the below Fig.



2.16. Simulation diagram:

The simulation diagram is a term can be defined as the connection diagram by using analogue tools to describe the differential equation of the mathematical model for any system. The elements are used in simulation diagram can be shown by:

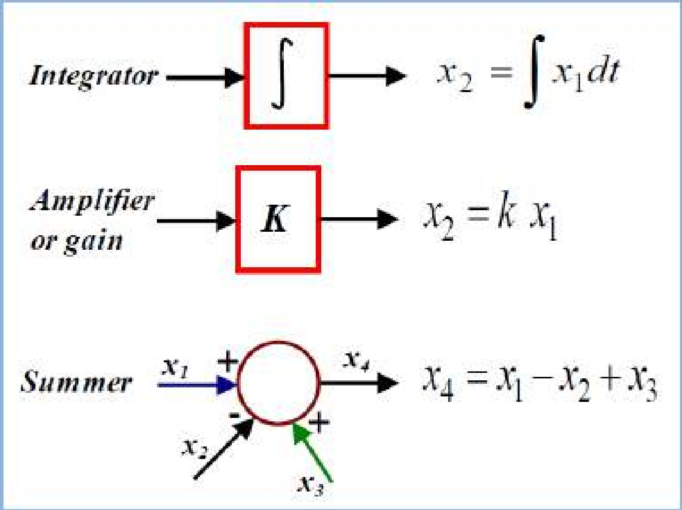


Fig.2.18. Elements used in a simulation diagram.

One of the methods used to obtain a simulation diagram includes the following steps:

1)Start with differential equation.

2)On the left side of the equation put the highest-order derivative of the dependent variable. A first-order or higher-order derivative of the input may appear in the equation. In this case the highest-order derivative of the input is also placed on the left side of the equation. All other terms are put on the right side.

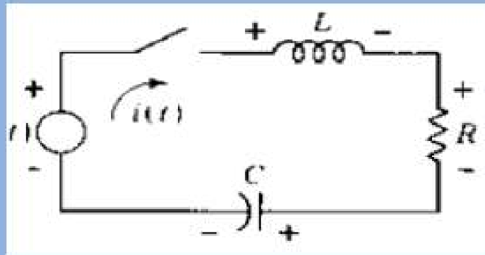
3)Start the diagram by assuming that the signal, represented by the terms on the left side of the equation, is available. Then integrate it as many times as needed to obtain all the lower-order derivatives. It may be necessary to add a summer in the simulation diagram to obtain the dependent variable explicitly.

4) Complete the diagram by feeding back the approximate outputs of the integrators to a summer to generate the original signal of step 2. Include the input function if it is required.

Example (17): Draw the simulation diagram for the series RLC circuit of Fig. below in which the output is the voltage across the capacitor.

Solution:

For the series RLC shown above ,the applied voltage equal to the sum of the voltage drops when the switch is closed.



$$V_l + V_c + V_R = e$$

$$l\frac{di}{dt} + \frac{1}{c}\int idt + iR = e$$

Step1. When y=V_c and u=e we get

$$LCY^{\cdot\cdot} + RCY^{\cdot} + Y = u$$

Step 2. Rearrange the terms to the form

$$Y^{\cdot\cdot} = \frac{1}{LC}u - \frac{R}{L}Y^{\cdot} - \frac{1}{LC}Y$$

$$Y^{\cdot\cdot} = bu - aY^{\cdot} - bY$$

Where

a= R/L and b=1/LC

step3.the signal $\ddot{Y}$ is integrated twice as shown in simulink implementation (Fig.a)

step4. The complete block diagram can be illustrated as in Fig.b.

the state variables are often selected as the output of the integrators in the simulation diagram.

In this case they are :

$$y = x_1,$$

$$\dot{y} = x_2 = \dot{x}_1$$

$$\ddot{y} = \dot{x}_2$$

The state space representation of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u = AX + Bu$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u = cX + Du$$

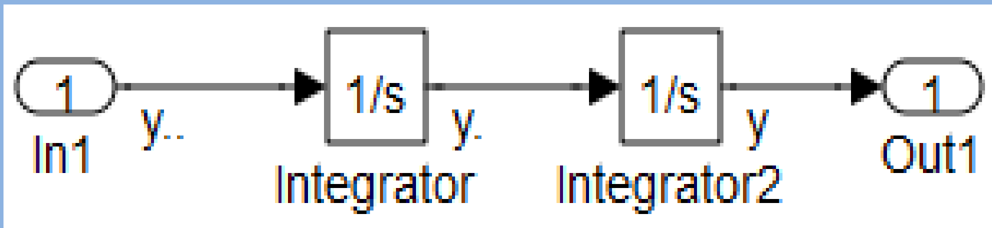


Fig.2.19. Simulink implementation

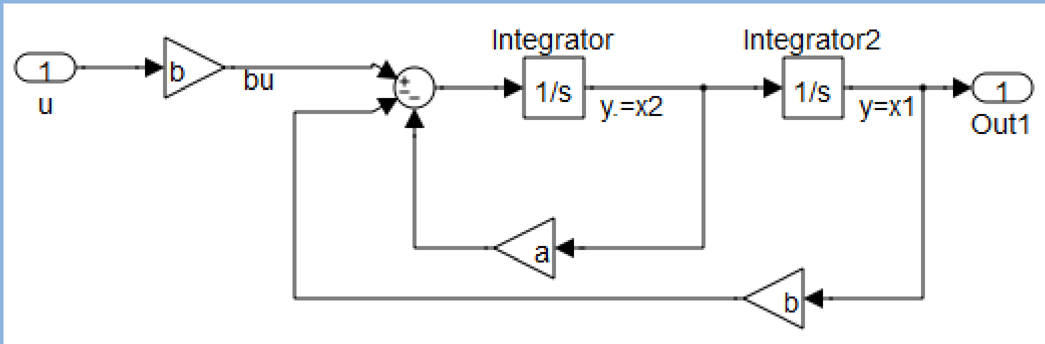


Fig.2.20. Simulink implementation of system state space

Example (18): Draw the simulation diagram that explain the differential equation below: $y^{\cdots}+2y^{\cdot\cdot}-5y^{\cdot}-7y=5\sin(u)$

$$y = x_1 \Rightarrow \dot{y} = \dot{x}_1 = x_2$$

Solution : $x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} = x_3$

$$\ddot{y} = x_3 \Rightarrow \dot{x}_3 = \dddot{y} = 5\sin(u) - 2\ddot{y} + 5\dot{y} + 7y$$

From the above eqn's can be draw the following Simulink diagram can be obtained:

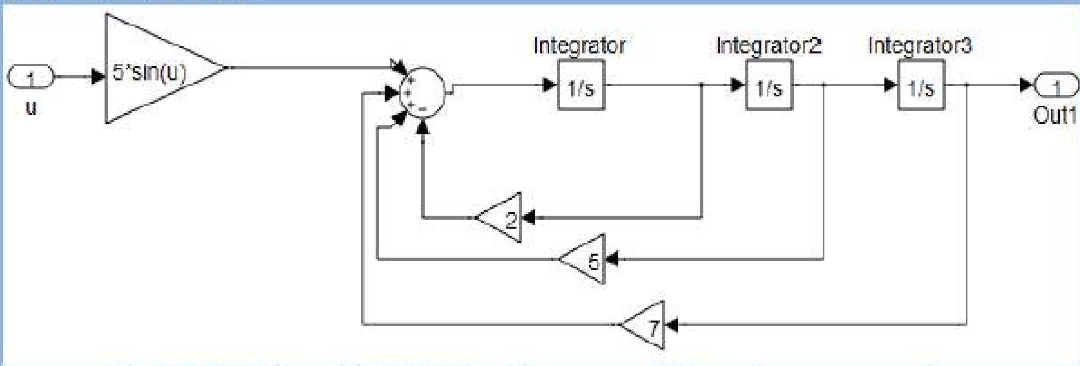


Fig.2.21. Simulink block diagram of the given example

H.W.(1): For the positional servomechanism obtain the closed-loop T.F. for the positional servomechanism shown Fig.2.22. Assume that the in-put and output of the system are:

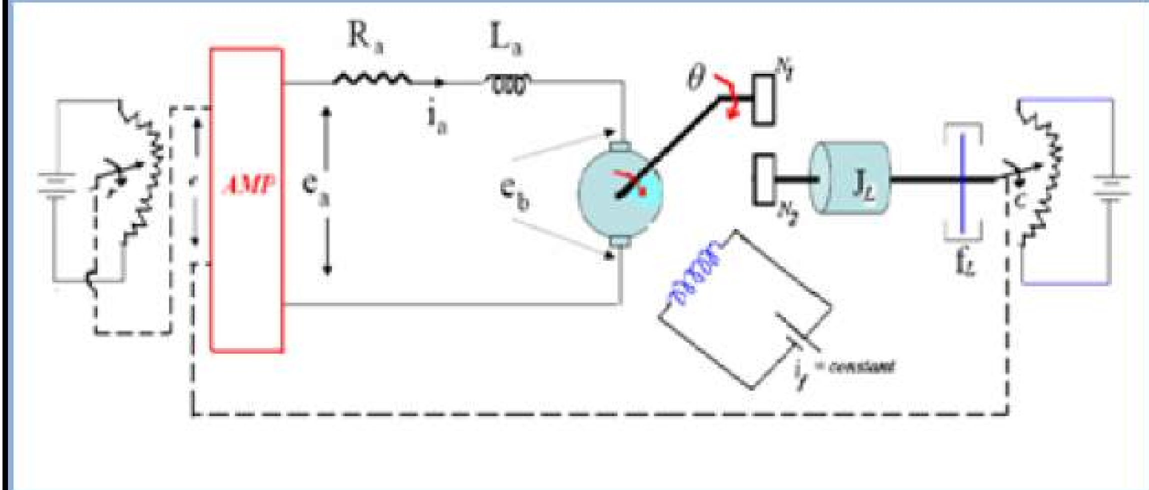


Fig.2.22. Circuit Diagram of the positional servomechanism.

input shaft position and the output shaft

r = reference input shaft, radian

c = output shaft, radian

θ = motor shaft, radian

k_1 = gain of potentiometer error detector = 24/π volt/rad

k_p = amplifier gain = 10

k_b = back emf const.= 5.5*10-2 volts-sec/rad

K = motor torque constant = 6*10-5 Ib-ft-sec2

R_a = 0.2 Ω

L_a = negligible

J_m = 1*10-3 Ib-ft-sec2

f_m = negligble

J_l = 4.4*10-3 Ib-ft-sec2

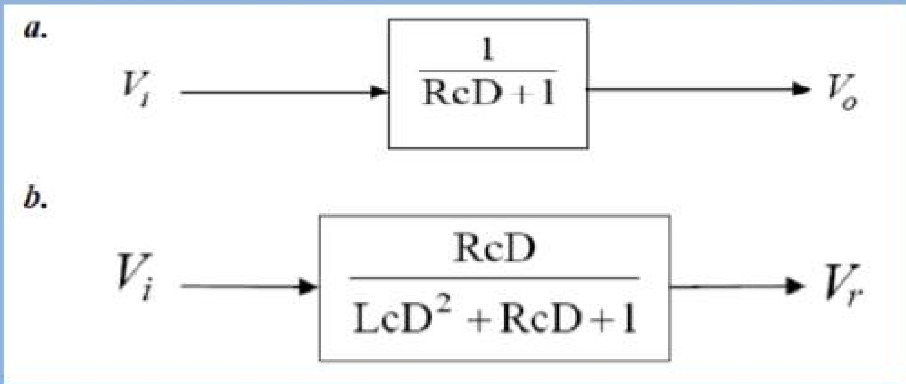
f_L = 4*10-2 Ib-ft/rad/sec

n = gear ratio N1/N2=1/10

Hint $J = J_m+n_2J_l$, $f = f_m+n_2f_L$

Answer $\dfrac{\theta(s)}{E_a(s)} = \dfrac{0.72}{s(0.13s+1)}$

H.W.(2): Find the circuits diagram for every transfer function below:



University of Anbar
College of Engineering
Dept. of Electrical Engineering

Lecture No. Three

Block Diagrams

Processing

This lecture discusses the following topics :

3.1. Introduction.

3.2. Symbols used in block diagrams (B.D).

3.3. Block Diagram reduction rules.

3.4. Variables in the Block diagram.

3.5. The Block Diagram Components.

3.6. Solved Problems

3.1. Introduction

The representation of physical components by blocks is shown in Lecture two for each block the transfer function provides the dynamical mathematical relationship between the input and output quantities. Also, lecture one describe the concept of feedback, which is used to achieve a better response of a control system to a command input. Now, the control systems represents by block diagrams. The blocks represent the functions performed rather than the components of the system.

3.2. Symbols used in block diagrams(B.D):

a. *Block:* the transfer function(TF) of the system element is placed in the block symbolized by,

Fig.3.1. Block diagram of a TF

b. *Summing points:* The operation of addition or subtraction is performed by this system element and symbolized by.

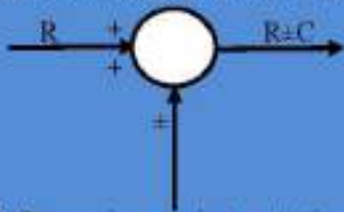


Fig.3.2.Summing point notation

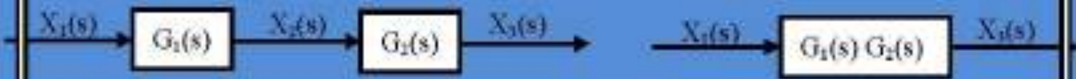
c. *Take off point:* This operation is used to provide a dual input (i/p) or output(o/p) to a system element and it is represented by,

Fig.3.3.Take off notation

d. *Direction arrows:* this symbol defines a unidirectional flow of the signal

Fig.3.4.Arrow notation

e. *Cascaded rule:* If two (or more) no load blocks in the same direction the output as follow:



$$G_1 = \frac{X_2(s)}{X_1(s)} \quad and \quad G_{12} = \frac{X_3(s)}{X_2(s)}; \rightarrow G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)X_3(s)}{X_1(s)X_2(s)} = G_1(s)G_2(s)$$

Fig.3.5. Cascaded rule

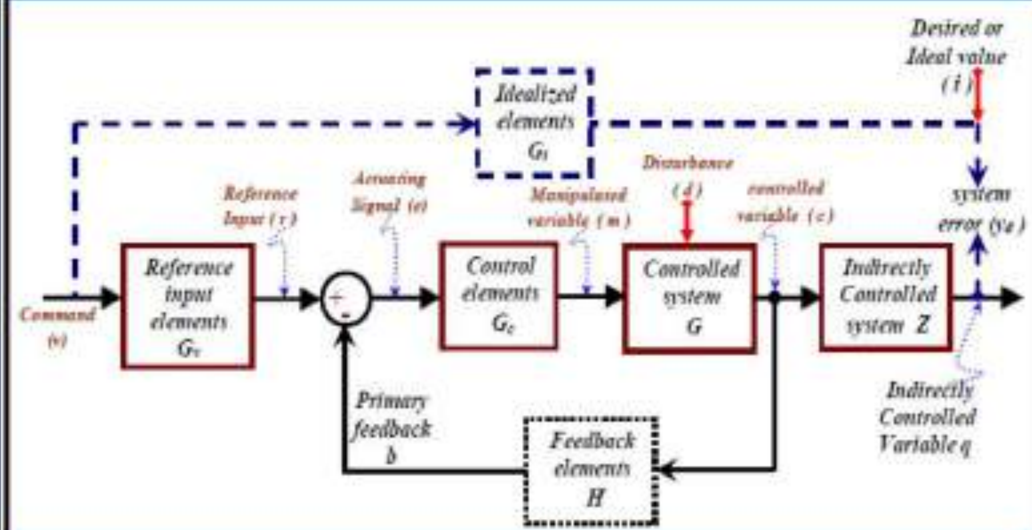
3.3. Variables in the Block diagram:

For the block diagram shown in Fig. below can be define the following term, that is represents the standard control system term in the representation of physical system in block diagram form:

Command (v): is the input that is established by some means external to, and independent of, the feedback control system.

Reference input (r): is derived from the command and is the actual signal input to the system.

Controlled variable (c): is the quantity that is directly measured and controlled. It is the output of the controlled system.



Primary feedback (b): is a signal that is a function of the controlled variable and that is compared with the reference input to obtain the actuating signal.

Actuating signal (e): is obtained from a comparison measuring device and is the reference input minus the primary feedback. This signal, usually at a low energy level, is the input to the control elements that produce the manipulated variable.

Fig.3.6. closed loop control system with all signals

Manipulated variable(m): is the quantity obtained from the control elements that is applied to the controlled system. The manipulated variable is generally at a higher energy level than the actuating signal and may also be modified in form.

Indirectly controlled variable (q): is the output quantity that is related through the indirectly controlled system to the controlled variable. It is outside the closed loop and is not directly measured for control.

Ultimately controlled variable: is a general term that refers to the indirectly controlled variable. In the absence of the indirectly controlled variable, it refers to the controlled variable.

Ideal value i: is the value of the ultimately controlled variable that would result from an idealized system operating with the same command as the actual system.

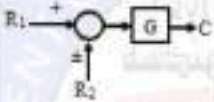
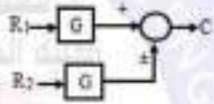
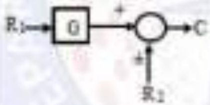
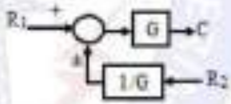
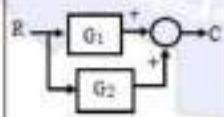
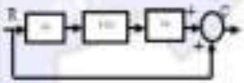
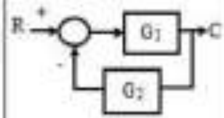
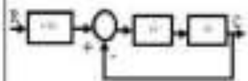
System error (ye): is the ideal value minus the value of the ultimately controlled variable.

Disturbance (d): is the unwanted signal that tends to affect the controlled variable. The disturbance may be introduced into the system at many places.

3.4. Block Diagram reduction rules:

The rules are using in the block diagram reduction to reduce the complex block diagram as single block diagram between input and output.

Table 3.1. Rules of Block Diagram Reduction

Transformation	B.D	Equivalent B.D	Equation(T.F)
Moving summing point beyond a block			$\frac{C}{R_1 \pm R_2} = G$
Moving summing point a head of a block			$C = R_1 G \pm R_2$
Moving block to the forward path			$C = R_1(G_1 \pm G_2)$
Moving F/B to the forward path			$\frac{C}{R} = \frac{G_1}{1 + G_1 G_2}$

3.5. The Block Diagram Components:

Reference input elements Gv: produce a signal (**r**) proportional to the command.

Control elements Gc: produce the manipulated variable m from the actuating signal.

Controlled system G: is the device that is to be controlled. This is frequently a high-power element.

Feedback element (H): produces the primary feedback b from the controlled variable. This is generally a proportionality device but may also modify the characteristics of the controlled variable.

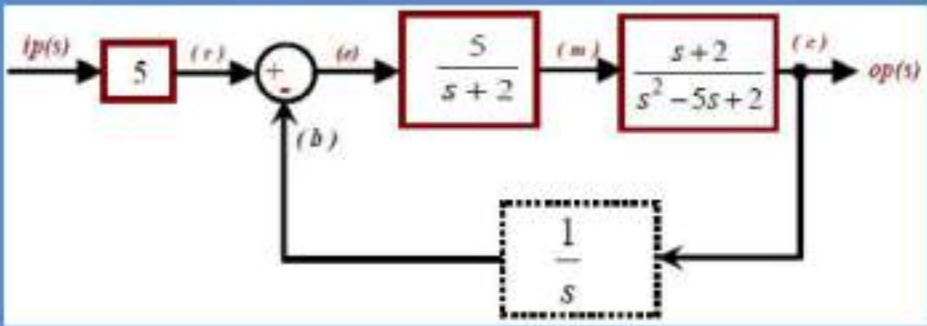
Indirectly controlled system Z: relates the indirectly controlled variable **(q)** to the controlled quantity **(c)**. This component is outside the feedback loop.

Idealized system Gi: is one whose performance is agreed upon to define the relationship between the ideal value and the command. This is often called the model or desired system.

Disturbance element N: denotes the functional relationship between the variable representing the disturbance and its effect on the control system.

3.6. Solved Problems

Prob.1. For the block diagram shown below find the overall transfer function ip/op?



Solution:

By Appling the rules of B.D. reduction can be get:

by cascaded no load blocks can be reduce the forward direction:

$$G_1(s) = \frac{5}{s+2}$$

$$G_2(s) = \frac{s+2}{s^2+5s+2}$$

The feed forward TF is:

$$G_1(s) * G_2(s) = \frac{5s+10}{s^3-3s^2-8s+4}$$

By using negative feedback rule we get

$$G(s) = \frac{G_1(s) * G_2(s)}{1+G_1(s) * G_2(s) H(s)}, H(s) = 1/s$$

or

$$G(s) = \frac{G_f(s)}{1+G_f(s) H(s)}, G_f(s) = G_1(s) G_2(s)$$

The final TF will be:

$$G(s) = \frac{25s^2 + 50s}{s^4 - 3s^3 - 8s^2 + 9s + 10}$$

Prob.2. **(Position Control System):**

Fig.3.7. below shows a simplified block diagram of an angular position control system. The reference selector and the sensor, which produce the reference input $R=\theta_R$ and the controlled output position $c=\theta_o$, respectively, consist of rotational potentiometers. The combination of these units represents a rotational comparison unit that generates the actuating signal **E** for the position control system, as shown in Fig.3.8 ,where $k\theta$, in volts per radian, is the potentiometer sensitivity constant. The symbolic comparator for

this systemis shown in Fig.3.9. The transfer function of the motor-generator control is obtained by writing the equations for the schematic diagram shown in Fig.3.8. This figure shows a dc motor that has a constant field excitation and drives an inertia and friction load. The armature voltage for the motor is furnished by the generator, which is driven at constant speed by a prime mover. The generator voltage e_g is determined by the voltage ef applied to the generator field. The generator is acting as a power amplifier for the signal voltage e_f .

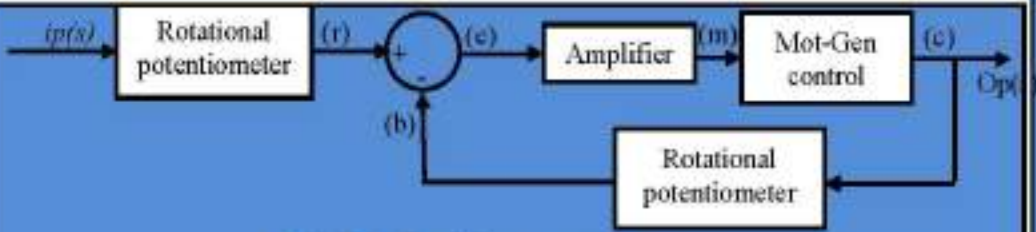


Fig.3.7. Position control system

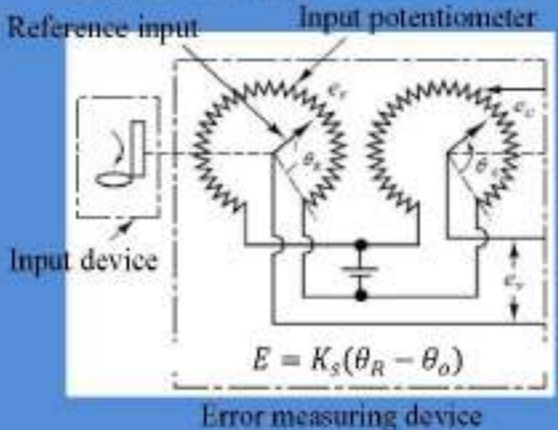


Fig.3.8. Rotational position comparison

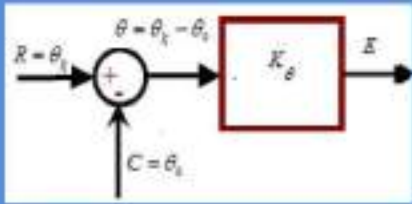


Fig.3.9. its block diagram representation

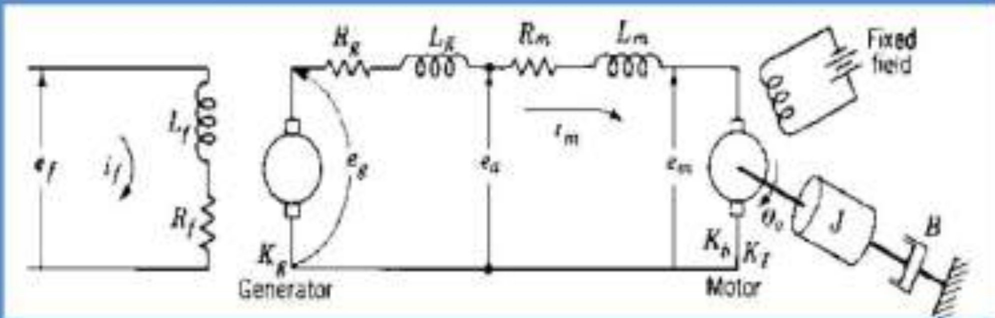


Fig.3.10. Motor-generator control

The equations that described the system in Fig.3.10. can be written as:

$$e_f = L_f D i_f + R_f i_f$$

(3.1)

$$e_g = K_g i_f$$

(3.2)

$$e_g - e_m = (L_g + L_m) D i_m + (R_g + R_m) i_m$$

(3.3)

$$e_m = K_b D\theta_0$$

(3.4)

$T = K_T\, i_m = JD^2\theta_o + BD\theta_o$

(3.5)

According to the eq's(3.1-3.5) can be draw the block diagram of the system in Fig.3.10. same as the Fig.3.11;

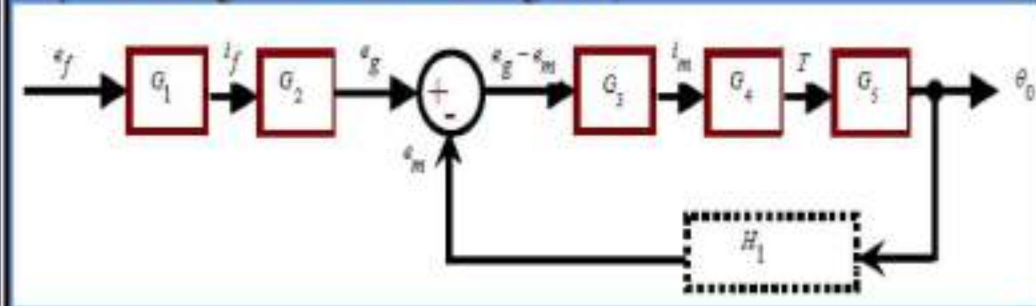


Fig.3.11. Block Diagram of the Motor-generator control system

The transfer functions of each block, as determined in terms of the pertinent Laplace transforms, are as follows:

$$G_1(s) = \frac{I_f(s)}{E_f(s)} = \frac{1/R_f}{1+(L_f/R_f)S} = \frac{1/R_f}{1+T_f S}$$

$$G_2(s) = \frac{E_L(s)}{I_f(s)} = K_g$$

$$G_3(s) = \frac{I_m(s)}{E_g(s) - E_m(s)} = \frac{1/(R_g + R_m)}{1+[(L_g + L_m)/(R_g + R_m)]S} = \frac{1/R_{gm}}{1+(L_{gm}/R_{gm})S}$$

$$G_4(s) = \frac{T(s)}{I_m(s)} = K_T \quad ;$$

$$G_5(s) = \frac{\Theta_0(s)}{T(s)} = \frac{1/B}{s[(1+(J/B)s]} = \frac{1/B}{s(1+T_v s)}$$

$H_1(s) = \frac{E_m(s)}{\Theta_0(s)} = K_b s$; The block diagram can be simplified, as shown in Fig.3.12, by combining the blocks in cascade. The block diagram is further simplified, to get the final transfer finction of vthe system as follow:

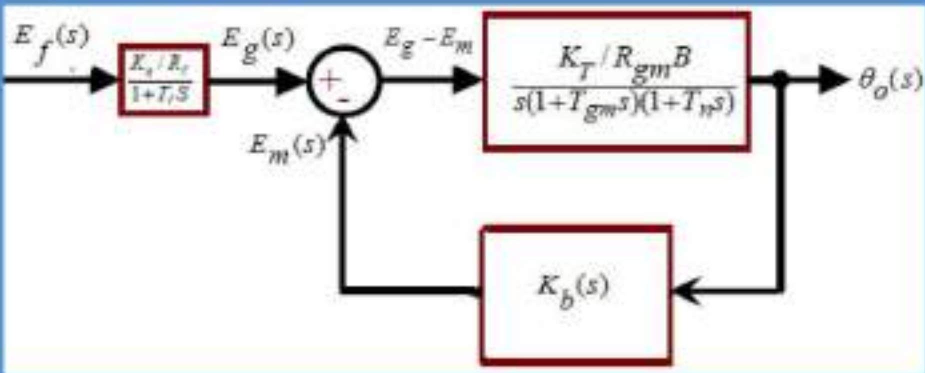
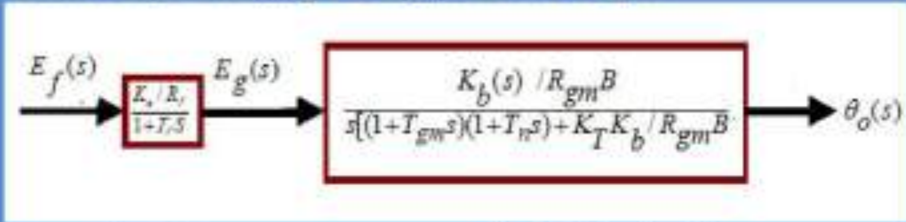


Fig.3.12. simplified block diagram

The next step is to apply the negative rule to get:



$$G_x(s)=\frac{\Theta_o(s)}{E_f(s)}=\frac{K_T K_g / R_f R_{gm} B}{s(1+T_f s)[(1+K_T K_b / BR_{gm})+(T_{gm}+T_n)s+T_{gm}T_n s^2]}$$

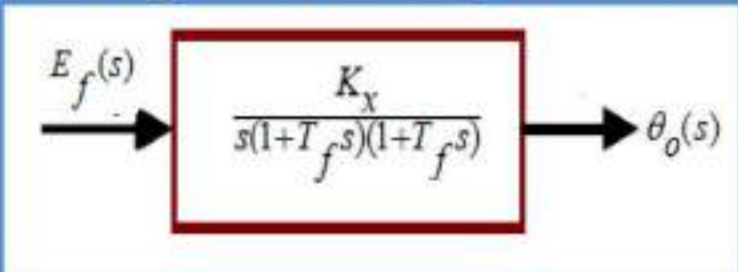
$$G_x(s)=\frac{\Theta_o(s)}{E_f(s)}\approx\frac{K_x}{s(1+T_f s)(1+T_f s)}$$

Where

$$K_x = \frac{K_T K_g}{R_f(BR_{gm} + K_T K_b)}; T_f = \frac{L_f}{R_f} \text{ and } T_m = \frac{JR_{gm}}{BR_{gm} + K_T K_b}$$

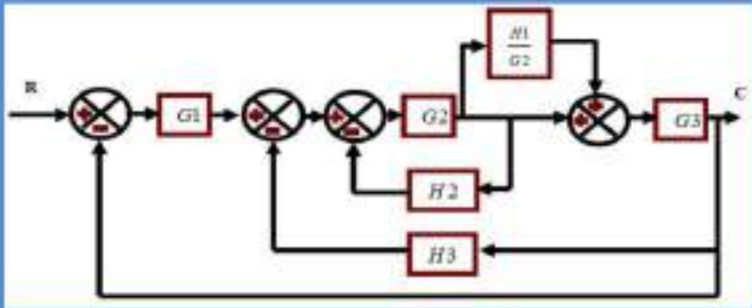
$$K_x = \frac{K_g}{R_f K_b}$$

The final block diagram of the this example will be as follows:



H.W. Find the Simulation Diagram for the transfer function $G_x(s)$?

Prob.3. Simplified the block diagram for the system shown below .

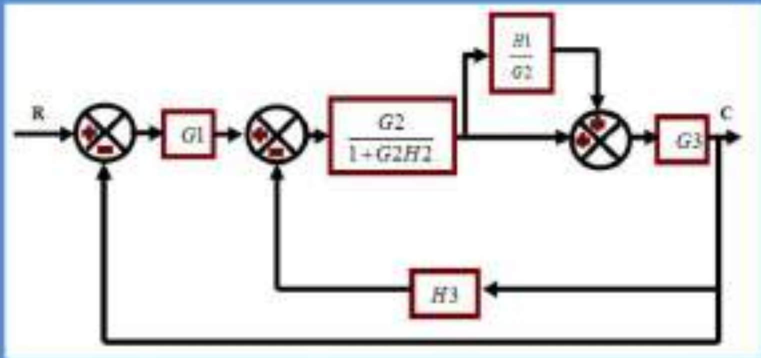


Solution:

We can notice the following

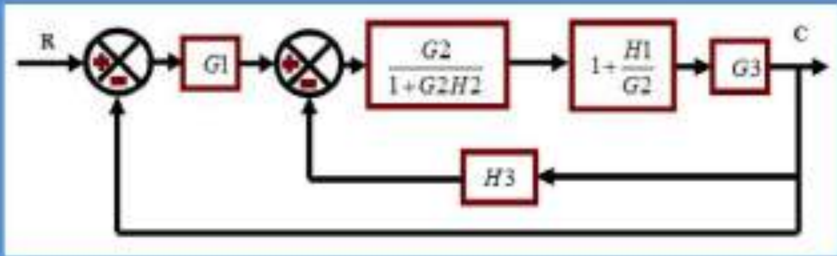
1.G2 and H2 is standard negative feedback system and the transfer function will be as:

$$Gt1 = \frac{G2}{1+G2H2}$$



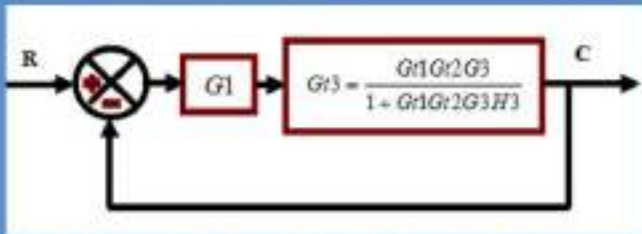
2. The transfer function of the parallel combiation of H1/G2 and unity forward path will be as :

$$Gt2 = 1 + \frac{H1}{G2}$$

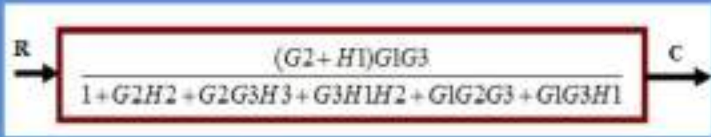


3.Gt1 ,Gt2 ,G3 and H3 combination is a standard negative feedback control system and its transfer function will be:

$$Gt3 = \frac{Gt1Gt2G3}{1+Gt1Gt2G3H3}$$



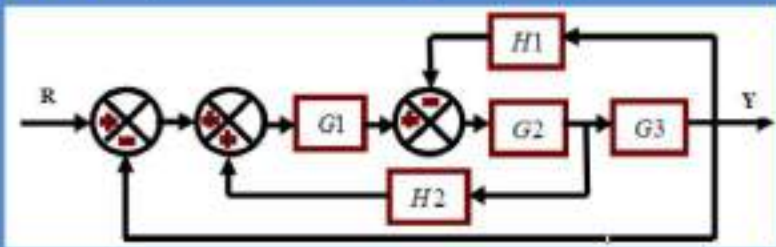
4.Gt3,G1 and unity feedback control system is a standard negative feedback its transfer function will as in the follwing block diagram



The final transfer function of the system is:

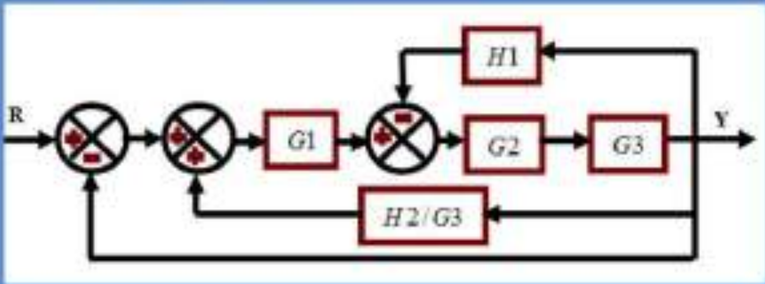
$$\frac{C}{R}=\frac{G1G3(G2+H1)}{1+G2H2+G2G3H3+G3H1H2+G1G2G3+G1G3H1}$$

Example 5. Simplified the following block diagram below:

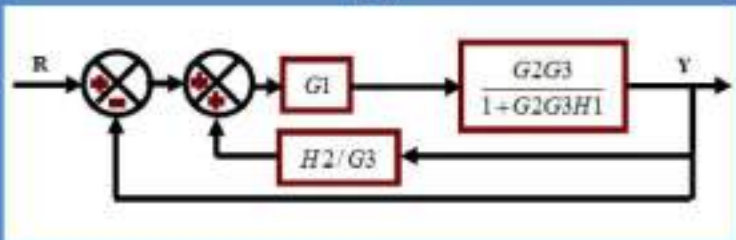


(a)

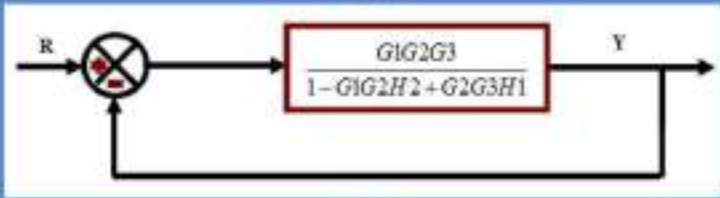
By using the standard rules for reduction can be get the overall all T.F by the steps (b-e), That is shown below:



(b)



(c)



(d)

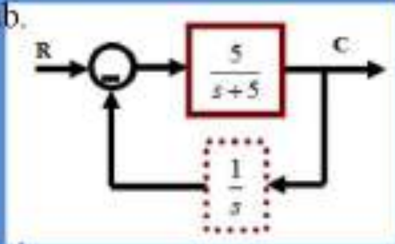
(e)

Fig.3.13(b-e): Steps for the solution of reduction the B.D. in Fig.3.13.a.

The final transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G1G2G3}{1 - G1G2H2 + G2G3H1 + G1G2G3}$$

Prob.4. For Matlab :Find the overall transfer function(C(s)/R(s)) of the following control system using Matlab.



Solution:

MATLAB provides the "feedback" function

to calculate the overall (closed -loop) transfer function

as shown by the following m-file.

% The feedback command calculates the closed-loop transfer

% function for a given forward transfer function G(s)

% and feedback transfer function H(s)

% Type "help feedback" for more information

% Example 1:

% Define a forward transfer function

% using the tf command

% System = tf (numerator, denominator) Transfer function: 5/(s +5)

G = tf([5],[1 5])

% Define a feedback transfer function H(s)= 1/s

% using the tf command

H =tf([1],[1 0], % System = tf (numerator, denominator) Transfer function: 1/s

% For a non-unity, negative feedback system the

% closed loop transfer function is

cltf = feedback (G,H,-1)

%Transfer function:5s / (s^2 +5s+ 5)

% For positive feedback use "feedback (G,H,1)"

5s / (s^2 +5s -5)

% The forward transfer function is calculated by multiplying

% G(s) and H(s) ; G *H : Transfer function: 5 / (s^2+ 5s); The

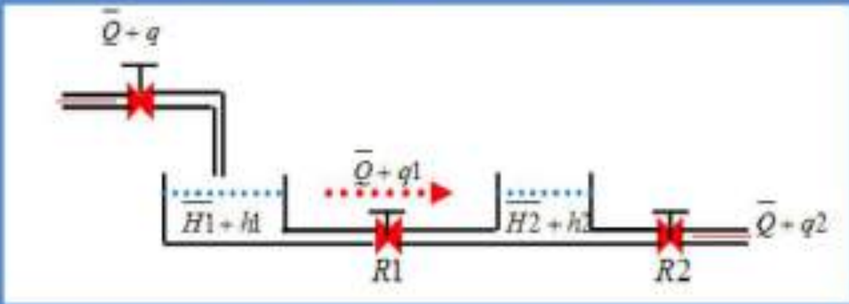
program:

G = tf([5],[1 5])

H =tf([1],[1 0]

cltf = feedback (G,H,-1)

Prob.5. Construct the block diagram for the system shown below(q,q1,q2,h1,h2) are changes from steady state.



$$q-q1=c1\frac{dh1}{dt}$$

$$q1=\frac{h1-h2}{R1}$$

$$q1-q2=c2\frac{dh2}{dt}$$

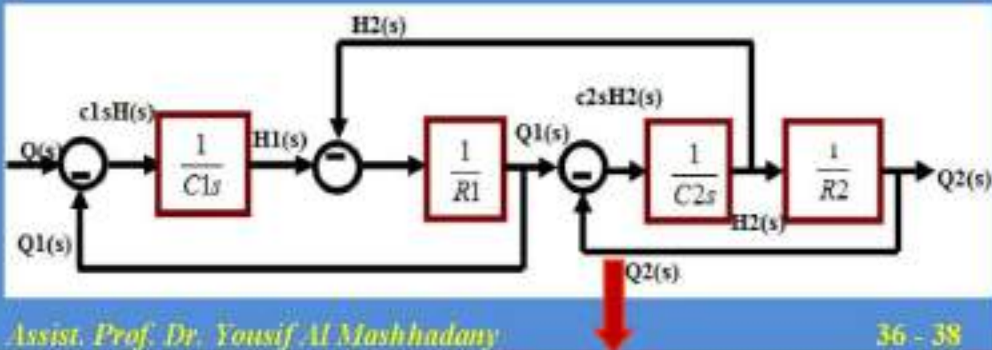
$q2 = \frac{h2}{R2}$; In Laplace transform

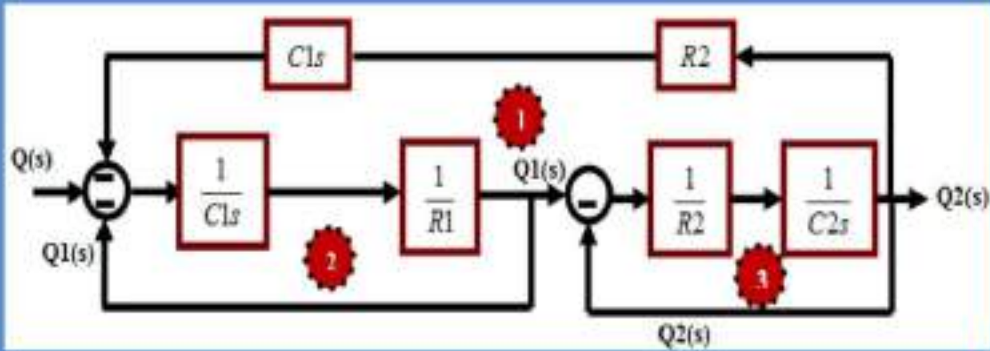
$Q(s) - Q1(s) = c1sH1(s)$;

$Q1(s) = \frac{H1(s) - H2(s)}{R1}$

$Q1(s) - Q2(s) = c2sH2(s)$

$Q2(s) = \frac{H2(s)}{R2}$



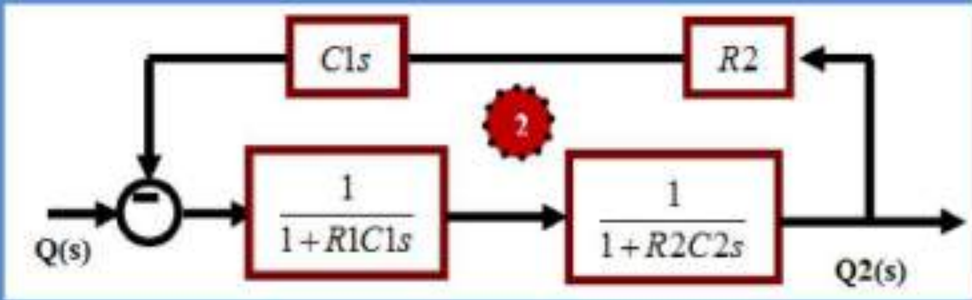


Loop 3:

$$\frac{Q2(s)}{Q1(s)} = \frac{(1/R2)(1/C2s)}{1+(1/R2)(1/C2s)} = \frac{1}{1+C2R2s}$$

Loop 2

$$\frac{Q(s)}{Q1(s)} = \frac{(1/R1)(1/C1s)}{1+(1/R1)(1/C1s)} = \frac{1}{1+C1R1s}$$



Loop 1.

$$\frac{Q2(s)}{Q1(s)}=\frac{(1/R1C1s)(1/R2C2s)}{1+(1/R1C1s)(1/R2C2s)C1R2s}$$

$$\frac{Q2(s)}{Q1(s)}=\frac{1}{C1C2R1R2s^2+(C1R1+C2R2+R2C1)s+1}$$